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Abstract. We study the homogenization limit of the space-heterogeneous renewal equation by means of the two-scale convergence theory. We prove that the homogenized limit satisfies an equation involving non-local terms, which are the consequence of the oscillations in the birth and death terms. The numerical approximation of the homogenized equation via the two-scale limit can give an alternative way for the numerical study of the solution of the limiting problem.

1. Introduction

Several mathematical models aim to describe cell dynamics, i.e. a process by which a parent cell divides into two or more daughter cells and then, at the end if its life cycle, it dies (see, for example, [2, 3, 4, 12]).

The first PDE model describing such a phenomenon is the well-known McKendrick model [7], which has first been introduced in the context of epidemiology, and then it has been used for modelling cell cultures evolution by von Foerster [19].

This equation has been widely mathematically studied and has been used as starting point for more elaborate models in biology and epidemiology, we refer to [11, 18] for an overview on the main mathematical properties of the model and examples of development.

The independent variables in the McKendrick model are time and age. Hence, it does not take into account possible spatial heterogeneities of the various terms appearing in it. However, there are several situations in which the cell evolution is influenced by the local properties of the host medium (or substrate).

In the literature, many studies have been concerned with determining the factors influencing growth, development, fission and death of bacteria. Some main factors are well known, such as the pH of the substrate, its temperature, the availability of chemical nutrients. It is indeed well known that these features have a major impact on cell metabolism (among the vast bibliography on the subject, see, for example, [5, 17, 8, 10, 13, 14, 20]).

In this article we investigate, from the mathematical viewpoint, how local heterogeneities in the properties of the substrate can modify the behaviour of the cell population with respect to the averaged (homogenized) case. In particular, we will consider a periodic spatial structure with high spatial frequency. It could mean, for example, that we are able to take into account the local temperature variability in the substrate, or variations in the presence of micronutrients. Our approach is based on the two-scale convergence theory, first proposed by Gabriel Nguetseng [9] and then developed by Grégoire Allaire [1].

Our analysis shows that the homogenized limit satisfies an equation involving non-local terms. This behaviour is coherent with a phenomenon observed by Luc Tartar in the context of ordinary differential equations with oscillating terms [15]. In our case, both the oscillations in the equation and in the boundary conditions have an influence on the emergence of non-local effects in the homogenization procedure.

The structure of the paper is the following. We first describe the problem in Section 2 and then, after a concise description of the two-scale convergence theory (Section 3), we study the two-scale limit in Section 4.

2. The space-dependent renewal equation

The renewal equation is one of the standard models which describe the vital dynamics of a population of cells, structured by age, with birth and death phenomena (for a deep study on this equation, see [11]).

In this article, we suppose that the vital dynamics is influenced by the properties of the substrate. In what follows, we suppose that the space variable belongs to the interval $X = (0, 1)$ and that the time horizon $\tau \in \mathbb{R}_+^*$ of the problem is strictly positive and finite.

The age-dependent spatial density is described by a function $u : \mathbb{R}_+^* \times (0, \tau) \times X \to \mathbb{R}_+^*$, defined a.e.. Here and in what follows, $a \in \mathbb{R}_+^*$ is the age variable, $t \in [0, \tau)$ denotes the time variable and $x \in X$ denotes the space variable.

We suppose that the speeds of the birth and death processes have a local dependence in space (for example, we suppose that the division process is mediated by some properties of a heterogeneous substrate). We introduce the heterogeneous (in space) age-dependent birth rate $\sigma_b : \mathbb{R}_+^* \times X \to \mathbb{R}_+^*$ and the heterogeneous age-dependent death rate $\sigma_d : \mathbb{R}_+^* \times X \to \mathbb{R}_+^*$.

Under the previous assumptions, the evolution of the density u satisfies the following equation:

$$\partial_t u(a, t, x) + \partial_a u(a, t, x) = -\sigma_d(a, x) u(a, t, x), \quad (a, t, x) \in \mathbb{R}_+^* \times (0, \tau) \times X \tag{2.1}$$

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with boundary condition

$$u(0,t,x) = \int_0^{+\infty} \sigma_b(\alpha,x)u(t,\alpha,x)\,\mathrm{d}\alpha, \qquad (t,x) \in (0,\tau)\times X \tag{2.2}$$

and initial condition

$$u(a,0,x) = u_{\mathrm{in}}, \qquad (a,x) \in \mathbb{R}_+^* \times X. \tag{2.3}$$

The well-posedness of the problem has been studied by several authors, see [11]. We provide here an alternative proof, suitable for our goals, which guarantees existence and uniqueness of the solution in a bounded time interval.

Theorem 1. *Let σ_d and σ_b be two non-negative functions of class $L^\infty(X;L^2(\mathbb{R}_+^*))$. Then, if the initial condition $u_{\mathrm{in}} \in L^\infty(X;L^2(\mathbb{R}_+^*))$ and $u_{\mathrm{in}} \geq 0$ for a.e. $(a,x) \in \mathbb{R}_+^* \times X$, then the initial value problem 2.1–2.2–2.3 has one and only one non-negative strong solution $u \in L^2(\mathbb{R}_+^*; L^\infty((0,\tau)\times X))$.*

Proof. We write the initial value problem 2.1–2.2–2.3 in integral form, by using the method of characteristics. We deduce that, for a.e. $(a,t,x) \in \mathbb{R}_+^* \times (0,\tau) \times X$,

$$\begin{aligned} u(a,t,x) &= \mathbb{1}_{t<a} u_{\mathrm{in}}(a-t,x) \exp\left(-\int_0^t \sigma_d(s+a-t,x)\,\mathrm{d}s\right) \\ &+ \mathbb{1}_{t>a}\left[\int_0^{+\infty} \sigma_b(\alpha,x)u(\alpha,t-a,x)\,\mathrm{d}\alpha\right] \exp\left(-\int_{t-a}^t \sigma_d(s+a-t,x)\,\mathrm{d}s\right). \end{aligned} \tag{2.4}$$

If we denote

$$F(u_{\mathrm{in}},\sigma_d) := \mathbb{1}_{t<a} u_{\mathrm{in}}(a-t,x) \exp\left(-\int_0^t \sigma_d(s+a-t,x)\,\mathrm{d}s\right) \tag{2.5}$$

and, for any $h \in L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*))$,

$$Th := \mathbb{1}_{t>a}\left[\int_0^{+\infty} \sigma_b(\alpha,x)h(\alpha,t-a,x)\,\mathrm{d}\alpha\right] \exp\left(-\int_{t-a}^t \sigma_d(s+a-t,x)\,\mathrm{d}s\right),$$

Equation 2.4 can be seen as a fixed-point problem:

$$u = F(u_{\mathrm{in}},\sigma_d) + Tu. \tag{2.6}$$

In what follows, we will show that

$$u := \sum_{n=0}^{+\infty} T^n F(u_{\mathrm{in}},\sigma_d) \tag{2.7}$$

is well-defined and the unique solution of Equation 2.4 in $L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*))$, which is embedded in $L^2(\mathbb{R}_+^* \times (0,\tau) \times X)$ because the set $(0,\tau)\times X$ has finite Lebesgue measure in $\mathbb{R}^2$.

The space $L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*))$ is a Banach space with norm

$$\|g\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))} = \operatorname*{ess\,sup}_{(t,x)\in(0,\tau)\times X} \left(\int_0^{+\infty} |g(a,t,x)|^2\,\mathrm{d}a\right)^{1/2}.$$

It is clear that $T : L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*)) \to L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*))$ and that it is linear.

It is moreover bounded because

$$\begin{aligned} &\|Th\|^2_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))} = \operatorname*{ess\,sup}_{(t,x)\in(0,\tau)\times X} \left(\int_0^{+\infty} |Th(a,t,x)|^2\,\mathrm{d}a\right) \\ &= \operatorname*{ess\,sup}_{(t,x)\in(0,\tau)\times X} \int_0^{+\infty} \left(\mathbb{1}_{t>a}\left[\int_{\mathbb{R}_+^*} \sigma_b(\alpha,x)h(\alpha,t-a,x)\,\mathrm{d}\alpha\right]\right. \\ &\left.\exp\left(-\int_{t-a}^t \sigma_d(s+a-t,x)\,\mathrm{d}s\right)\right)^2 \mathrm{d}a \\ &\leq \|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))} \operatorname*{ess\,sup}_{(t,x)\in(0,\tau)\times X} \int_0^t \int_{\mathbb{R}_+^*} h^2(\alpha,t-a,x)\,\mathrm{d}\alpha\,\mathrm{d}a \\ &= \|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))} \|h\|^2_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\tau, \end{aligned}$$

that is

$$\|T\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))} \leq \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}\sqrt{\tau}.$$

We can generalize the previous computations by studying the bound on the iterated operator T^n with respect to the norm $\|\cdot\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))}$. We deduce that, for any $n\in\mathbb{N}^*$ and for any $h\in L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))$, when $t\in(0,\tau)$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in X}\int_0^{+\infty}|T^n h(a,t,x)|^2\,\mathrm{d}a\\
&\le \|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\int_0^t \|T^{n-1}h(\cdot\,,t-a,\,x)\|^2_{L^2(\mathbb{R}_+^*)}\,\mathrm{d}a\\
&= \|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\int_0^t \|T^{n-1}h(\cdot\,,t_1,\,x)\|^2_{L^2(\mathbb{R}_+^*)}\,\mathrm{d}t_1\\
&\le \|\sigma_b\|^{2n}_{L^\infty(X;L^2(\mathbb{R}_+^*))}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\|h(\cdot\,,t_n,x))\|^2_{L^2(\mathbb{R}_+^*)}\,\mathrm{d}t_n\\
&\le \frac{1}{n!}\left(\|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\right)^n\|h\|^2_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\tau^n.
\end{aligned}$$

By passing to the supremum for $t\in(0,\tau)$ in both sides of the previous inequality and noticing that the right-hand side is independent on t, we have that

$$\|T^n\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))}\le\frac{1}{\sqrt{n!}}\left(\|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}\sqrt{\tau}\right)^n$$

for all $n\in\mathbb{N}^*$.

Because of 2.5, we have that

$$\|F(u_{\mathrm{in}},\sigma_d)\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\le\|u_{\mathrm{in}}\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}. \tag{2.8}$$

Consequently,

$$\begin{aligned}
&\sum_{n=0}^{+\infty}\|T^nF(u_{\mathrm{in}},\sigma_d)\|^2_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\\
&\qquad\le\|F(u_{\mathrm{in}},\sigma_d)\|^2_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\exp\left(\|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\tau\right).
\end{aligned}$$

Therefore, u defined as in (2.7) exists and is norm-bounded:

$$\begin{aligned}
\|u\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}&\le\|F(u_{\mathrm{in}},\sigma_d)\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\exp\left(\|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\frac{\tau}{2}\right)\\
&\le\|u_{\mathrm{in}}\|_{L^2(\mathbb{R}_+^*;L^\infty(X))}\exp\left(\|\sigma_b\|^2_{L^\infty(X;L^2(\mathbb{R}_+^*))}\frac{\tau}{2}\right).
\end{aligned}\tag{2.9}$$

Moreover, u solves the integral problem 2.6 because

$$\begin{aligned}
u=\sum_{n=0}^{+\infty}T^nF(u_{\mathrm{in}},\sigma_d)&=F(u_{\mathrm{in}},\sigma_d)+\sum_{n=1}^{+\infty}T^nF(u_{\mathrm{in}},\sigma_d)\\
&=F(u_{\mathrm{in}},\sigma_d)+T\sum_{n=0}^{+\infty}T^nF(u_{\mathrm{in}},\sigma_d)=F(u_{\mathrm{in}},\sigma_d)+Tu.
\end{aligned}$$

The solution is moreover unique. Indeed, suppose that there exist two distinct (in a.e. sense) solutions u_1 and u_2 of 2.6. Then, by finite induction, their difference u_1-u_2 is such that, for any $n\in\mathbb{N}^*$

$$u_1-u_2=T(u_1-u_2)=T^2(u_1-u_2)=\cdots=T^n(u_1-u_2).$$

By passing to the norm

$$\begin{aligned}
&\|u_1-u_2\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\\
&\qquad=\|T^n\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))}\|u_1-u_2\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\\
&\qquad\le\frac{1}{\sqrt{n!}}\left(\|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}\sqrt{\tau}\right)^n\|u_1-u_2\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}\to 0
\end{aligned}$$

as $n\to+\infty$. But u_1 and u_2 are distinct by hypothesis. Hence we have a contradiction and so the solution is unique. □

3. Basic concepts of two-scale convergence

The concept of two-scale convergence has been introduced by Gabriel Nguetseng [9] and developed by Grégoire Allaire [1]. Its definition is the following.

Definition 1. *Let X be a domain of $\mathbb{R}^d$ and $Y = (0,1)^d$. Denote with $C_{\mathrm{per}}(Y)$ the space of continuous functions on Y which are Y-periodic. A family of functions $z^\varepsilon(x) \subset L^2(X)$ two-scale converges to a limit $z^0(x,y) \in L^2(X \times Y)$ if, for any test function $\psi(x,y) \in L^2(X; C_{\mathrm{per}}(Y))$, we have*

$$\lim_{\varepsilon\to 0} \int_X z^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_X \int_Y z^0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

The following compactness result is crucial for using the two-scale convergence theory.

Theorem 2. *Let X be a domain of $\mathbb{R}^d$ and $Y = (0,1)^d$. Let $z^\varepsilon(x) \subset L^2(X)$ be a uniformly bounded family of functions such that*

$$\|z^\varepsilon\|_{L^2(X)} \le C$$

where the constant C is independent of ε. Then, there exists a subsequence extracted from z^ε (still denoted z^ε) such that z^ε two-scale converges to some limit $z^0(x,y) \in L^2(X \times Y)$.

Another important property of the two-scale limit is given by the following proposition.

Proposition 3. *Let X be a domain of $\mathbb{R}^d$ and $Y = (0,1)^d$. Let $z^\varepsilon \subset L^2(X)$ a family of functions which two-scale converges to a limit $z^0 \in L^2(X \times Y)$. Then $z^\varepsilon(x)$ converges to*

$$\langle z\rangle(x) = \int_Y z^0(x,y)\,\mathrm{d}y$$

weakly in $L^2(X)$, that is

$$\lim_{\varepsilon\to 0} \int_X z^\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_X \varphi(x) \int_Y z^0(x,y)\,\mathrm{d}y\,\mathrm{d}x \qquad \textit{for all } \varphi \in L^2(X).$$

Hence, the two-scale convergence, which is given in terms of test functions (see Definition 1), is a form of weak convergence which implies the standard weak convergence, here in L^2. However, the following result gives a sufficient condition for improving this weak-type convergence.

Proposition 4. *Let X be a domain of $\mathbb{R}^d$ and $Y = (0,1)^d$. Let $z^\varepsilon(x)$ be a family such that it two-scale converges to $z^0(x,y)$. Then*

$$\lim_{\varepsilon\to 0} \|z^\varepsilon\|_{L^2(X)} \ge \left\|z^0\right\|_{L^2(X\times Y)} \ge \|\overline{z}\|_{L^2(X)}$$

where $\overline{z}(x)$ is the weak L^2-limit of the family $z^\varepsilon(x)$. Moreover, if

$$\lim_{\varepsilon\to 0} \|z^\varepsilon\|_{L^2(X)} = \left\|z^0\right\|_{L^2(X\times Y)} \tag{3.1}$$

and if the two-scale limit $z^0(x,y) \in L^2(X; C_{per}(Y))$, then the following strong two-scale convergence holds

$$\lim_{\varepsilon\to 0} \left\| z^\varepsilon(\cdot) - z^0\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^2(X)} = 0.$$

4. Two-scale homogenization of the renewal equation

We reformulate here the initial-value problem 2.1–2.2–2.3 by introducing a scale parameter $\varepsilon \in \mathbb{R}_+^*$. The parameter $\varepsilon > 0$ represents the heterogeneity length scale of the problem. The smaller the value of the parameter ε, the smaller the period of the spatial oscillations in σ^ε and $u_{\mathrm{in}}^\varepsilon$.

4.1. The initial-boundary value problem.

Our goal is to study the two-scale limit, as $\varepsilon \to 0^+$, of the following problem:

$$\partial_t u^\varepsilon(a,t,x) + \partial_a u^\varepsilon(a,t,x) = -\sigma_d^\varepsilon(a,x)u^\varepsilon(a,t,x), \qquad (a,t,x) \in \mathbb{R}_+^* \times (0,\tau) \times X \tag{4.1}$$

with boundary conditions

$$u^\varepsilon(t,0,x) = \int_0^{+\infty} \sigma_b^\varepsilon(\alpha,x)u^\varepsilon(\alpha,t,x)\,\mathrm{d}\alpha, \qquad (t,x) \in (0,\tau) \times X \tag{4.2}$$

and initial conditions

$$u^\varepsilon(0,a,x) = u_{\mathrm{in}}^\varepsilon(a,x), \qquad (a,x) \in \mathbb{R}_+^* \times X. \tag{4.3}$$

The coefficients and data in 4.1 are of the form

$$\sigma_d^\varepsilon(a,x) := \sigma_d\left(a, x, \frac{x}{\varepsilon}\right), \quad \sigma_b^\varepsilon(a,x) := \sigma_b\left(a, x, \frac{x}{\varepsilon}\right)$$

and

$$u_{\mathrm{in}}^\varepsilon(a,x) := u_{\mathrm{in}}\left(a, x, \frac{x}{\varepsilon}\right).$$

In the asymptotics as $\varepsilon \to 0$, we derive the corresponding homogenized equation. In this setting, the variable x is a parameter. Note that the regularity of the birth rate σ_b and of the death rate σ_d, as well as the composition with continuous functions like the exponential, makes them suitable as test functions in the two-scale convergence (see Definition 1).

4.2. **Hypotheses on the birth and death rates.** Let $Y = (0,1)$ and denote with $C_{\mathrm{per}}(Y)$ the space of continuous functions on Y which are Y-periodic.

We suppose that σ_d and σ_b are locally periodic with respect to the fast oscillations in the space variable and that

$$\sigma_d(a,x,y) \text{ and } \sigma_b(a,x,y) \in L^2(\mathbb{R}_+^*; C(X; C_{\mathrm{per}}(Y))) \cap L^\infty((0,\tau)\times X; L^2(\mathbb{R}_+^*)). \tag{4.4}$$

Moreover, we suppose that there exist two strictly positive constants $\sigma_{\min}$ and $\sigma_{\max}$ such that

$$\sigma_{\max} \geq \sigma_d(a,x,y) \geq \sigma_{\min} \quad \text{for all } (x,y) \in X \times Y \text{ and for a.e. } a \in \mathbb{R}_+^* \tag{4.5}$$

and

$$\sigma_{\max} \geq \sigma_b(a,x,y) \geq \sigma_{\min} \quad \text{for all } (x,y) \in X \times Y \text{ and for a.e. } a \in \mathbb{R}_+^*. \tag{4.6}$$

Remark 1. *Let $\varphi \in L^2(\mathbb{R}_+^*; C(X; C_{per}(Y)))$. The hypotheses on σ_d and σ_b guarantee that $\sigma_d\varphi \in L^2(\mathbb{R}_+^*; C(X; C_{per}(Y)))$ and $\sigma_b\varphi \in L^2(\mathbb{R}_+^*; C(X; C_{per}(Y)))$.*

4.3. **The homogenization procedure.** Clearly, Theorem 1 guarantees the existence and the uniqueness of the solution of 4.1-4.2-4.3 for all $\varepsilon > 0$. Because of the boundedness of $(0,\tau)\times X$, we can deduce that $u^\varepsilon \in L^2((0,\tau)\times\mathbb{R}_+^*\times X)$ for all $\varepsilon > 0$.

Consider now the family $(u^\varepsilon)_{\varepsilon>0}$ of solutions to the initial-boundary value problem 4.1-4.2-4.3 and study the limit of the family as $\varepsilon \to 0^+$. By analogy with a remark by Tartar [16, 15], we can expect, for this system, the existence of memory effects induced by the two-scale homogenization procedure.

Denote with $L^2_{\mathrm{per}}(Y)$ the set of L^2 functions on Y which are periodic in Y. For any $g \in L^\infty(Y)$, we introduce, as in [6], the linear operator

$$\mathcal{L}_g\, h := gh - \langle gh\rangle \quad \forall h \in L^2_{\mathrm{per}}(Y),$$

We underline that the operator $\mathcal{L}_g$ is bounded in $L^2_{\mathrm{per}}(Y)$ because

$$\|\mathcal{L}_g h\|^2_{L^2_{\mathrm{per}}(Y)} = \int_Y |g(y)h(y) - \langle gh\rangle|^2 \,\mathrm{d}y = \int_Y |g(y)h(y)|^2 \,\mathrm{d}y - \langle gh\rangle^2$$

and, by applying the Cauchy-Schwarz inequality,

$$|\langle gh\rangle| = \left|\int_Y g(y)h(y)\,\mathrm{d}y\right| \leq \left(\int_Y |g(y)h(y)|^2 \,\mathrm{d}y\right)^{1/2}.$$

We are now ready to prove our homogenization result for the evolution 4.1-4.2-4.3 in the framework of the two-scale convergence theory.

Theorem 5. *Let $u^\varepsilon(t,x)$ be the solution of the evolution problem 4.1-4.2-4.3, with a ε-dependent initial condition $u^\varepsilon_{\mathrm{in}} \in L^\infty(X; L^2(\mathbb{R}_+^*))$ which two-scale converges to $u_{\mathrm{in}}(a,x,y) \in L^2(\mathbb{R}_+^*\times X\times Y)$. Suppose moreover that the birth and death rates satisfy the hypotheses of Subsection 4.2. Then, up to a subsequence,*

$$u^\varepsilon \rightharpoonup u_{\mathrm{hom}} \quad \textit{weakly in } L^2(\mathbb{R}_+^*\times(0,\tau)\times X)$$

and $u_{\mathrm{hom}}(t,x)$ solves the following integro-differential equation

$$\begin{cases} \partial_t u_{\mathrm{hom}}(a,t,x) + \partial_a u_{\mathrm{hom}}(a,t,x) = \\ \qquad\qquad -\langle\sigma_d\rangle(a,x)u_{\mathrm{hom}}(a,t,x) - \langle\sigma_d \displaystyle\sum_{n=0}^{+\infty}\mathcal{S}^n\mathcal{Q}(u_{\mathrm{in}},\sigma_d)\rangle(a,t,x) \\ u_{\mathrm{hom}}(t,0,x) = \displaystyle\int_0^{+\infty}\left[\langle\sigma_b\rangle(\alpha,x)u_{\mathrm{hom}}(\alpha,t,x) + \langle\sigma_b\sum_{n=0}^{+\infty}\mathcal{S}^n\mathcal{Q}(u_{\mathrm{in}},\sigma_d)\rangle(\alpha,t,x)\right]\mathrm{d}\alpha, \\ u_{\mathrm{hom}}(0,a,x) = \langle u_{\mathrm{in}}\rangle(a,x), \end{cases} \tag{4.7}$$

where

$$\begin{aligned} \mathcal{S}h :=& \mathbb{1}_{t>a} e^{-\int_{t-a}^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_0^{+\infty}\Big[\sigma_b(\alpha,x,y)h(\alpha,t-a,x,y) \\ &- \int_Y \sigma_b(\alpha,x,y)h(\alpha,t-a,x,y)\,\mathrm{d}y\Big]\,\mathrm{d}\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta}\int_Y \sigma_d(a,x,y)h(a+s-t,s,x,y)\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

for any $h \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$ *and*

$$\mathcal{Q}(u_{\rm in}, \sigma_d) := \mathbb{1}_{t<a} e^{-\int_0^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \left[u_{\rm in}(a-t,x,y) - \int_Y u_{\rm in}(a-t,x,y)\,\mathrm{d}y \right].$$

Remark 2. *Note that the two-scale homogenization limit exhibits two memory terms, both in the equation and in the boundary conditions at* $a = 0$*. Moreover,* $u_{\rm in}^\varepsilon$ *two-scale converges, up to a subsequence, to* $\langle u_{\rm in}\rangle$ *by hypothesis (which is coherent with the regularity of the family* $u_{\rm in}^\varepsilon$*).*

Proof. The proof is based on the integral form of the evolution equation 4.1-4.2-4.3:

$$\begin{aligned} u^\varepsilon(a,t,x) &= \mathbb{1}_{t<a} u_{\rm in}\left(a-t, x, \frac{x}{\varepsilon}\right) \exp\left(-\int_0^t \sigma_d\left(s+a-t, x, \frac{x}{\varepsilon}\right)\,\mathrm{d}s\right) \\ &+ \mathbb{1}_{t>a}\left[\int_0^{+\infty} \sigma_b\left(\alpha, x, \frac{x}{\varepsilon}\right) u^\varepsilon(\alpha, t-a, x)\,\mathrm{d}\alpha\right] \exp\left(-\int_{t-a}^t \sigma_d\left(s+a-t, x, \frac{x}{\varepsilon}\right)\,\mathrm{d}s\right). \end{aligned} \tag{4.8}$$

We moreover note that the estimate 2.9 is satisfied by all member of the family $(u^\varepsilon)_{\varepsilon>0}$ and that the estimate is uniform in ε, because all quantities involved in 2.9 are ε-independent and keeping in mind that $X = (0,1)$, we have indeed that

$$\begin{aligned} \|u^\varepsilon\|_{L^2(\mathbb{R}_+^* \times (0,\tau)\times X)} &\le \tau \|u^\varepsilon\|_{L^2(\mathbb{R}_+^*; L^\infty((0,\tau)\times X))} \\ &\le \|u_{\rm in}\|_{L^2(\mathbb{R}_+^*; L^\infty(X))} \tau \exp\left(\sigma_{\max}^2 \frac{\tau}{2}\right) =: C < +\infty \end{aligned} \tag{4.9}$$

for all $\varepsilon > 0$. Therefore, by Theorem 2, there exists a subsequence, still denoted u^ε, which two-scale converges to a function $u^0 \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$, i.e.:

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\mathbb{R}_+^* \times (0,\tau)\times X} & u^\varepsilon\left(a,t,x,\frac{x}{\varepsilon}\right) \psi\left(a,t,x,\frac{x}{\varepsilon}\right)\,\mathrm{d}a\mathrm{d}t\mathrm{d}x \\ &= \int_{\mathbb{R}_+^* \times (0,\tau)\times X \times Y} u^0(a,t,x,y) \psi(a,t,x,y)\,\mathrm{d}a\mathrm{d}t\mathrm{d}x\mathrm{d}y, \end{aligned}$$

for any test-function ψ satisfying the regularity hypotheses of Definition 1.

We can hence deduce the following equality, in the sense of the two-scale limit and up to a subsequence :

$$\begin{aligned} u^0(a,t,x,y) &= \mathbb{1}_{t<a} u_{\rm in}(a-t,x,y) \exp\left(-\int_0^t \sigma_d(s+a-t,x,y)\,\mathrm{d}s\right) \\ &+ \mathbb{1}_{t>a}\left[\int_0^{+\infty} \sigma_b(\alpha,x,y) u(\alpha,t-a,x)\,\mathrm{d}\alpha\right] \exp\left(-\int_{t-a}^t \sigma_d(s+a-t,x,y)\,\mathrm{d}s\right). \end{aligned} \tag{4.10}$$

Consequently, the limit u^0 solves the two-scale evolution equation

$$\partial_t u^0(a,t,x,y) + \partial_a u^0(a,t,x,y) = -\sigma_d(a,x,y) u^0(a,t,x,y), \tag{4.11}$$

for any $(t,a,x,y) \in \mathbb{R}_+^* \times (0,\tau) \times X \times Y$ with boundary conditions :

$$u^0(t,0,x,y) = \int_0^{+\infty} \sigma_b(\alpha,x,y) u^0(\alpha,t,x,y)\,\mathrm{d}\alpha, \qquad (t,x,y) \in (0,\tau)\times X \times Y \tag{4.12}$$

and initial conditions

$$u^0(0,a,x,y) = u_{\rm in}(a,x,y), \qquad (a,x) \in \mathbb{R}_+^* \times X \times Y. \tag{4.13}$$

By Proposition 3, we deduce that the sequence u^ε converges weakly in $L^2(\mathbb{R}^+ \times\ (0,\tau)\times X \times Y)$ to

$$u_{\rm hom}(a,t,x) := \langle u^0\rangle(a,t,x).$$

We conclude our proof by deducing the equation satisfied by $u_{\rm hom}$. We decompose the two-scale limit into a homogeneous part, denoted $u_{\rm hom}$, and a remainder r, with zero mean over the periodic cell, i.e.

$$u^0(a,t,x,y) = u_{\rm hom}(a,t,x) + r(a,t,x,y) \quad \text{and} \quad \langle r\rangle = 0. \tag{4.14}$$

We then replace 4.14 into Equation 4.11, which governs the space-time evolution of the two-scale limit u^0. We obtain

$$\begin{aligned} \partial_t u_{\rm hom}(a,t,x) + \partial_a u_{\rm hom}(a,t,x) + \partial_t r(a,t,x,y) + \partial_a r(a,t,x,y) \\ = -\sigma_d(a,x,y) u_{\rm hom}(a,t,x) - \sigma_d(a,x,y) r(a,t,x,y) \end{aligned} \tag{4.15}$$

for $(a,t,x,y) \in \mathbb{R}_+^* \times (0,\tau) \times X \times Y$. Equations 4.12 and 4.13 become respectively

$$u_{\rm hom}(t,0,x) + r(t,0,x,y) = \int_0^{+\infty} \sigma_b(\alpha,x,y)[u_{\rm hom}(\alpha,t,x) + r(\alpha,t,x,y)]\,\mathrm{d}\alpha \tag{4.16}$$

and

$$u_{\rm hom}(0,a,x) + r(0,a,x,y) = u_{\rm in}(a,x,y). \tag{4.17}$$

We integrate Equation 4.15 over the periodicity cell Y, thus obtaining

$$\partial_t u_{\mathrm{hom}}(a,t,x) + \partial_a u_{\mathrm{hom}}(a,t,x) = -\langle \sigma_d \rangle (a,x) u_{\mathrm{hom}}(a,t,x) - \langle \sigma_d r \rangle (a,t,x). \tag{4.18}$$

On the other hand, if we integrate Equations 4.12 and 4.13 over the periodicity cell Y, we deduce

$$u_{\mathrm{hom}}\,(t,0,x) = \int_0^{+\infty} \Big[\langle \sigma_b \rangle (\alpha,x) u_{\mathrm{hom}}(\alpha,t,x) + \langle \sigma_b r \rangle (\alpha,t,x)\Big]\,\mathrm{d}\alpha, \tag{4.19}$$

and

$$u_{\mathrm{hom}}\,(0,a,x) = \langle u_{\mathrm{in}} \rangle (a,x). \tag{4.20}$$

By inserting 4.18, 4.19 and 4.20 respectively in 4.15, 4.16 and 4.17, we get the initial-boundary value problem for the remainder term:

$$\partial_t r(a,t,x,y) + \partial_a r(a,t,x,y) = -\big[\sigma_d(a,x,y) r(a,t,x,y) - \langle \sigma_d r \rangle (a,t,x)\big]. \tag{4.21}$$

The initial and the boundary conditions become respectively

$$\begin{aligned} r\,(t,0,x,y) &= \int_0^{+\infty} \left[\sigma_b(\alpha,x,y) r(\alpha,t,x,y) - \langle \sigma_b r \rangle (\alpha,t,x)\right]\,\mathrm{d}\alpha \\ &= \int_0^{+\infty} \mathcal{L}_{\sigma_b} r(\alpha,t,x,y)\,\mathrm{d}\alpha \end{aligned} \tag{4.22}$$

and initial conditions

$$r\,(0,a,x,y) = u_{\mathrm{in}}(a,x,y) - \langle u_{\mathrm{in}} \rangle (a,x) = \mathcal{L}_1 u_{\mathrm{in}}(a,x,y). \tag{4.23}$$

We have thus deduced the following coupled initial-boundary problems for the unknowns r and u_{hom}:

$$\begin{cases} \partial_t r(a,t,x,y) + \partial_a r(a,t,x,y) = \displaystyle\int_Y \sigma_d(a,x,y) r(a,t,x,y)\,\mathrm{d}y - \sigma_d(a,x,y) r(a,t,x,y) \\ r\,(t,0,x,y) = \displaystyle\int_0^{+\infty} \left[\sigma_b(\alpha,x,y) r(\alpha,t,x,y) - \int_Y \sigma_b(\alpha,x,y) r(\alpha,t,x,y)\,\mathrm{d}y\right]\,\mathrm{d}\alpha \\ r\,(0,a,x,y) = u_{\mathrm{in}}(a,x,y) - \langle u_{\mathrm{in}} \rangle (a,x) \end{cases} \tag{4.24}$$

and

$$\begin{cases} \partial_t u_{\mathrm{hom}}(a,t,x) + \partial_a u_{\mathrm{hom}}(a,t,x) = -\langle \sigma_d \rangle (a,x) u_{\mathrm{hom}}(a,t,x) - \langle \sigma_d r \rangle (a,t,x) \\ u_{\mathrm{hom}}\,(t,0,x) = \displaystyle\int_0^{+\infty} \Big[\langle \sigma_b \rangle (\alpha,x) u_{\mathrm{hom}}(\alpha,t,x) + \langle \sigma_b r \rangle (\alpha,t,x)\Big]\,\mathrm{d}\alpha, \\ u_{\mathrm{hom}}\,(0,a,x) = \langle u_{\mathrm{in}} \rangle (a,x). \end{cases} \tag{4.25}$$

Note that no term involving u_{hom} appears in problem 4.24. We write it in integral form, thus obtaining

$$\begin{aligned} r(a,t,x,y) &= \mathbb{1}_{t<a} e^{-\int_0^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \left[u_{\mathrm{in}}(a-t,x,y) - \int_Y u_{\mathrm{in}}(a-t,x,y)\,\mathrm{d}y\right] \\ &+ \mathbb{1}_{t>a} e^{-\int_{t-a}^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_0^{+\infty} \Big[\sigma_b(\alpha,x,y) r(\alpha,t-a,x,y) \\ &- \int_Y \sigma_b(\alpha,x,y) r(\alpha,t-a,x,y)\,\mathrm{d}y\Big]\,\mathrm{d}\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_Y \sigma_d(a,x,y) r(a+s-t,s,x,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

We then introduce the quantities

$$\mathcal{Q}(u_{\mathrm{in}},\sigma_d) := \mathbb{1}_{t<a} e^{-\int_0^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \left[u_{\mathrm{in}}(a-t,x,y) - \int_Y u_{\mathrm{in}}(a-t,x,y)\,\mathrm{d}y\right] \tag{4.26}$$

and, for any $h \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$,

$$\begin{aligned} \mathcal{S}h := \mathbb{1}_{t>a} e^{-\int_{t-a}^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} &\int_0^{+\infty} \Big[\sigma_b(\alpha,x,y) h(\alpha,t-a,x,y) \\ &- \int_Y \sigma_b(\alpha,x,y) h(\alpha,t-a,x,y)\,\mathrm{d}y\Big]\,\mathrm{d}\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_Y \sigma_d(a,x,y) h(a+s-t,s,x,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

By means of an argument similar to the proof of Theorem 1, we look for solutions of the fixed-point problem

$$r = \mathcal{Q}(u_{\rm in}, \sigma_d) + \mathcal{S} r. \tag{4.27}$$

We hence introduce the following *ansatz* on the structure of the solution:

$$r := \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\rm in}, \sigma_d), \tag{4.28}$$

and show that it gives the unique solution of 4.27 in $L^2(\mathbb{R}_+^* \times \mathbb{R}_+^* \times X \times Y)$.

We first remark that the linear operator $\mathcal{S}$ is well defined on $L^2(\mathbb{R}_+^*(0,\tau) \times X \times Y)$ and that its image belongs to $L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$.

Thanks to the triangular inequality and the standard Cauchy-Schwarz inequality, we indeed have that, for $h \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$,

$$\begin{aligned}
&\|\mathcal{S}h\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} \\
&\leq \left(2\|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\tau\right) \|h\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}.
\end{aligned}$$

This allows to deduce that

$$\|\mathcal{S}\|_{\mathcal{L}(L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y))} \leq \left(2\|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\tau\right)^{1/2},$$

i.e. $\mathcal{S}$ is bounded in $L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$ and hence continuous.

By following the same strategy, we obtain an estimate of the L^2-norm of the iterated operator $\mathcal{S}^n$. For any $n \in \mathbb{N}_*$ and for any $h \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$, we have that

$$\begin{aligned}
&\|\mathcal{S}^n h\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} \\
&\leq \left(2\|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\tau\right) \int_0^t \|\mathcal{S}^{n-1} h(\cdot, t-a,\, x)\|^2_{L^2(\mathbb{R}_+^*)} \mathrm{d}t_1 \\
&\leq \frac{1}{n!}\left(2\|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\tau\right)^n \|h\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}
\end{aligned}$$

i.e.

$$\begin{aligned}
&\|\mathcal{S}^n\|_{\mathcal{L}(L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y))} \\
&\leq \frac{\tau^{n/2}}{\sqrt{n!}} \left(2\|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\tau\right)^{n/2}
\end{aligned}$$

for all $n \in \mathbb{N}^*$.

Equation 4.26 implies that

$$\|\mathcal{Q}(u_{\rm in}, \sigma_d)\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} \leq 4\|u_{\rm in}\|^2{}_{L^2(\mathbb{R}_+^* \times X \times Y)}\tau.$$

Consequently,

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \|\mathcal{S}^n \mathcal{Q}(u_{\rm in}, \sigma_d)\|_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} \\
&\qquad \leq 2\|u_{\rm in}\|_{L^2(\mathbb{R}_+^* \times X \times Y)} \tau^{1/2} \exp\left(2\tau \|\sigma_b\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)} + \tau^2 \|\sigma_d\|^2_{L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)}\right).
\end{aligned}$$

Therefore, r exists and its norm is bounded. The same argument used in the existence and uniqueness proof for the solution of 2.6 shows that r is the unique solution of the integral formulation of 4.24. Hence, 4.25 can be written in the following form:

$$\begin{cases}
\partial_t u_{\rm hom}(a,t,x) + \partial_a u_{\rm hom}(a,t,x) = \\
\qquad\qquad -\langle \sigma_d \rangle(a,x) u_{\rm hom}(a,t,x) - \langle \sigma_d \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\rm in}, \sigma_d) \rangle(a,t,x) \\
u_{\rm hom}\,(t,0,x) = \displaystyle\int_0^{+\infty} \left[\langle \sigma_b \rangle(\alpha,x) u_{\rm hom}(\alpha,t,x) + \langle \sigma_b \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\rm in}, \sigma_d) \rangle(\alpha,t,x)\right] \mathrm{d}\alpha, \\
u_{\rm hom}\,(0,a,x) = \langle u_{\rm in} \rangle(a,x),
\end{cases} \tag{4.29}$$

where

$$\begin{aligned}\mathcal{S}h :=& \mathbb{1}_{t>a} e^{-\int_{t-a}^{t} \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_0^{+\infty} \Big[\sigma_b(\alpha,x,y) r(\alpha,t-a,x,y) \\ &- \int_Y \sigma_b(\alpha,x,y) r(\alpha,t-a,x,y)\,\mathrm{d}y\Big]\,\mathrm{d}\alpha \\ &+ \int_{(t-a)_+}^{t} e^{-\int_s^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \int_Y \sigma_d(a,x,y) r(a+s-t,s,x,y)\,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

for any $h \in L^2(\mathbb{R}_+^* \times (0,\tau) \times X \times Y)$ and

$$\mathcal{Q}(u_{\mathrm{in}},\sigma_d) := \mathbb{1}_{t<a} e^{-\int_0^t \sigma_d(a+\theta-t,x,y)\,\mathrm{d}\theta} \left[u_{\mathrm{in}}(a-t,x,y) - \int_Y u_{\mathrm{in}}(a-t,x,y)\,\mathrm{d}y\right].$$

□

Remark 3. *The result of Theorem 5 shows that the limit equation 4.7 has a much more complex structure than the two-scale limit problem 4.11-4.13. In particular, it contains memory terms. Such memory terms can be complicated to deal with numerically. Therefore, the two-scale limit problem can be used to numerically study the solution of the homogenized equation. The price to be paid is the introduction of an additional variable into the periodic cell, the advantage is that it allows to keep the local in-time character of the equation. In particular, a numerical strategy based on the two-scale limit does not require the entire time evolution of the solution to be handled at each time step.*

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