KINETIC BGK MODEL FOR A CROWD: CROWD CHARACTERIZED BY A STATE OF EQUILIBRIUM

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Abstract. In this paper we are interested in a dynamic description of the collective pedestrian motion based on the kinetic model of Bathnagar-Gross-Krook (BGK). In this model a pedestrians trend towards a state of equilibrium in a certain relaxation time is modeled. An approximation of the Maxwellian function that represents this equilibrium state is determined. A result of existence and uniqueness of the discrete velocity model is demonstrated. Thus the convergence of the solution to the solution of the continuous BGK equation is proven. Numerical tests are developed to validate the proposed mathematical model.

Keywords: Discrete kinetic theory, Crowd dynamics, BGK model, Semi-Lagrangian schemes

MSC 2010: 35A01, 35A02, 97M70, 97N40

1. **Introduction**

Mathematical representations of crowd motion from the microscopic to macroscopic scale have been an active field of study for the last three decades. An overview of the most important models at microscopic, macroscopic, and mesoscopic scale is reviewed in [1]. Indeed, the most popular crowd simulation models are the individual models, namely the heuristic rule-based models [2], mechanical models [3, 4, 5], and cellular automate [6], continuous models are based on fluid dynamics [7, 8, 9], and the kinetic (Gas-kinetic) models are intermediate models between the two discrete and continuous models [10, 11, 12]. Handerson was the first to apply this type of "kinetic gas" model to empirical pedestrian crowd data [13, 14].
In 2011 Bellomo et al. [15, 16, 17, 18, 19, 20] have developed the kinetic approach for crowds in a recent approach called, the kinetic theory for active particles. This approach considers the crowd as a complex system. The microscopic state of each particle is characterized by a geometric variable, position $\mathbf{x} = (x, y)$ and a mechanical variable velocity $\mathbf{v} = (v_x, v_y)$. In addition there is a microscopic state related to their socio-biological behavior, called activity, it noted u. The representation of the system is defined by a distribution function noted $f(t, \mathbf{x}, \mathbf{v}, u)$, where $f(t, \mathbf{x}, \mathbf{v}, u)d\mathbf{x}d\mathbf{v}$ represents the number of active pedestrians who at the moment t are in the elementary volume $[\mathbf{x}, \mathbf{x}+d\mathbf{x}]\times[\mathbf{v}, \mathbf{v}+d\mathbf{v}]$ and who have activity u. Pedestrian movements are governed by the partial derivative equation (PDE) of transport applied to f, Γ characterizing the different interactions between pedestrians and their environment

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v}, u)}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}, u) = \Gamma[f](t, \mathbf{x}). \tag{1.1}$$

In our previous work [21], we considered the model (1.1). The term $\Gamma[f](t, \mathbf{x})$ which models the interactions between pedestrians with various obstacles, is treated from a probabilistic point of view.

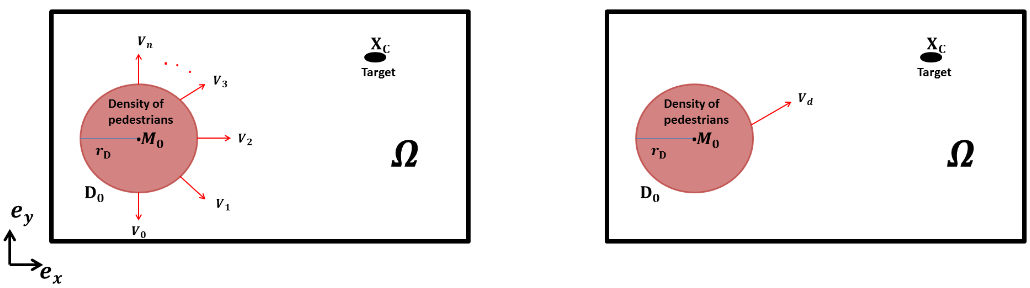
In this paper, we are interested in one of the simplest ways to model the term $\Gamma[f](t, \mathbf{x})$. It consists of describing a pedestrian tendency to equilibrium similar to the BGK operator which replaces the collision operator of the Boltzmann equation. Specifically, the case where the pedestrian system is characterized by an equilibrium configuration f_e and a relaxation term $\tau[\rho]$. In an emergency evacuation case, pedestrians try to achieve a desired velocity noted $\mathbf{v}_d$ to reach a target. $\tau[\rho]$ is the relaxation term describes the adaptation of the density f to the equilibrium density $f_{eq}(\mathbf{v})$. Therefore the interactions term takes the following simple form:

$$\Gamma[f](t, \mathbf{x}) = \frac{1}{\tau[\rho]} \left(f_{eq}(\mathbf{v} \rightarrow \mathbf{v}_d) - f \right). \tag{1.2}$$

This paper develops a special theory for pedestrian motion and these interactions namely deceleration avoidance. Consequently the BGK model (1.1), (1.2) proposed in this work does not use the assumptions of conservation of momentum and energy. Only the conservation of mass must be verified in our study. The equilibrium state function of pedestrians is developed and based on Henderson works [13, 14]. A mathematical framework for a theoretical study of the proposed model is determined. The rest of this paper is organized as follows: section 2 provides the discrete velocity model derived from the continuous BGK equation. This model describes the motion of pedestrians reaching an equilibrium configuration in a domain Ω. Then an approximation of the Maxwellian discrete density representing this state of equilibrium is presented. In section 3, a result of existence and uniqueness for the discrete velocity model is demonstrated. Then we prove the convergence of this solution to a solution of the continuous BGK equation. Section 4 is devoted to numerical simulations to validate the proposed model, and to show its ability to describe the main features of the dynamics of pedestrians.

2. Mathematical model

2.1. **Boltzmann equation: the BGK model for a crowd.** We consider a system composed of N particles (pedestrians) randomly distributed in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$.
At the moment $t = t_0$, the pedestrians are distributed in a disk D_0 of radius r_D and center $M_0(x_0, y_0)$. The initial overall density is then $\rho_0 = \frac{N}{\pi r_D^2} (ped/m^2)$.
This group of N pedestrians present in the domain Ω at the initial time t_0. They are in a normal traffic situation, i.e. they have the ability to direct themselves towards all possible directions. After, all pedestrians have the tendency of a comfortable destination noted $\mathbf{v}_d$ (state of equilibrium). The group of pedestrians wants to reach a target located at the point $\mathbf{x}_c$ (see Fig.1).



(a) Normal state (b) Equilibrium state

Figure 1. Density of pedestrians characterized by (a) normal state: pedestrians have the ability to direct towards n possible directions. (b) the equilibrium state: all pedestrians look for a certain destination comfortable noted $\mathbf{v}_d$ to achieve the target located at the point $\mathbf{x}_c$.

The state of the crowd is represented by the density $f(t, \mathbf{x}, \mathbf{v})$. They move with a velocity $\mathbf{v} \in D_{\mathbf{v}}$. The average crowd quantities obtained by integrating f in the velocity space $D_{\mathbf{v}}$:

- density:

$$\rho(t, \mathbf{x}) = \int_{D_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$

- average velocity:

$$\boldsymbol{\xi}(t, \mathbf{x}) = \frac{1}{\rho(t, \mathbf{x})} \int_{D_{\mathbf{v}}} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$

- total energy:

$$E = \frac{1}{2} \int_{D_{\mathbf{v}}} ||\mathbf{v}||^2 f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}.$$

The evolution of the particle density $f(t, \mathbf{x}, \mathbf{v})$ is described by the following equation:

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\tau[\rho]} \left(f_{eq}(\mathbf{v}) - f \right). \tag{2.1}$$

The coefficient τ can depend on the density $\rho(t, \mathbf{x})$, this term expresses that the distribution f would not go instantly to the desired velocity distribution f_{eq}, but would need some time called relaxation time τ. For reasons of simplicity, we assume that this relaxation term is a constant i.e. $\tau[\rho] = \tau$. According to this hypothesis, the model (2.1) takes the following form:

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\tau} \left(f_{eq}(\mathbf{v}) - f(t, \mathbf{x}, \mathbf{v}) \right). \tag{2.2}$$

We consider the model (2.1) with following initial data:

$$f(t = 0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \Omega, \ \mathbf{v} \in D_{\mathbf{v}}. \tag{2.3}$$

The system defined by the two equations (2.2), (2.3) represents the Bhatnagar-Gross-Krook model. In this paper it describes the temporal evolution of the distribution of particles (pedestrians). This model is less expensive than the Boltzmann equation because it is sufficient to update the macroscopic fields at each time step. On the other hand it provides qualitatively correct solutions for macroscopic moments. These two aspects, namely the relatively low cost of calculation and the correct description of the hydrodynamic limit, explain the interest in the BGK model during the last decades.

It also shares important features with Boltzmann's original equation, such as the following conservation laws:

$$\textbf{conservation of mass} \qquad \int_{D_{\mathbf{v}}} f_{eq}(\mathbf{v}) d\mathbf{v} = \int_{D_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}. \tag{2.4}$$

Thus, the BGK equation is a kinetic collision equation that takes into account only the overall effect of pedestrian interactions.

Remark 1. *The BGK model for pedestrian motion contains corrections due to interactions such as avoidance and deceleration. Therefore this model does not obey the conservation of momentum and energy. In our case, the only law of conservation which must be respected by the model is the conservation of the mass. It is expressed by the following equation:*

$$\int_{D_{\mathbf{v}}} f_{eq}(\mathbf{v}) d\mathbf{v} = \int_{D_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}. \tag{2.5}$$

2.2. **Maxwellian approximation: equilibrium density $f_{eq}(\mathbf{v})$.** In 1971 L.F.Henderson [13], suggested that the motion of people in a crowd represents a system similar to a gas molecule collection. Specifically, he suggested that the classical Maxwell-Boltzmann theory of a molecular system could also describe the velocity distribution of individuals.

Henderson measured the speed distribution function for 3 crowd categories: a crowd of university students walking from the library to the university, an adult crowd of all ages using a pedestrian crossing on a street, and children in a playground air. Analyses and estimates are made under certain numbers of assumptions about the crowd, namely, the movement is defined in every moment t by position (x, y) and the velocity $\mathbf{v} = (v_x, v_y)$. All the individuals in the crowd have the same mass.

The two figures 2 (a), (b) show that Henderson's empirical results agree with the classical Maxwell-Boltzmann theory. The distribution of the v_x component of the velocity is given by the following equation:

$$f_{eq}(v_x) = \frac{1}{N}\frac{dN_{v_x}}{dv_x} = \frac{1}{\sqrt{2\pi}v_m} exp\Big(-\frac{1}{2}\frac{v_x^2}{v_m^2}\Big), \tag{2.6}$$

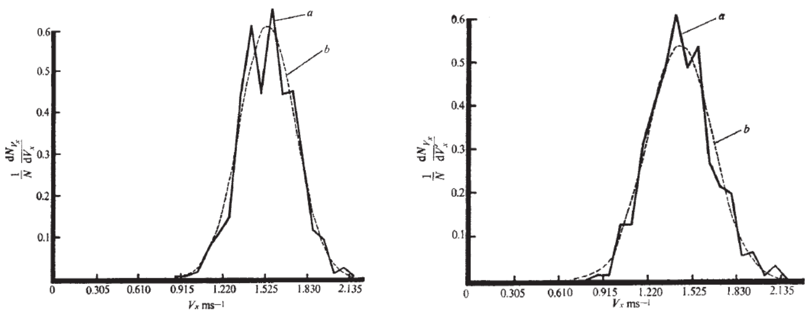
where,

- v_m is the square root of the average module value of speed.
- N the total number of pedestrians.
- N_{v_x} is the number of pedestrians with speed v_x.

In a similar way for the equation of the v_y component distribution of velocity, Henderson found the result for the distribution of $\mathbf{v} = (v_x, v_y)$, is given by the following equation :

$$f_{eq}(\mathbf{v}) = \frac{1}{N}\frac{dN_{\mathbf{v}}}{d\mathbf{v}} = \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{\mathbf{v}^2}{v_m^2}\Big), \tag{2.7}$$

where $N_{\mathbf{v}}$ is the number of pedestrians with velocity $\mathbf{v}$.



(a) (b)

Figure 2. The density function of the first component v_x of the speed (a) for 693 students walking outside the library at the University of Sydney. Curve a represents the measured distribution and curve b represents the Maxwell-Boltzmann distribution, $v_x = 1.53 m.s^{-1}$,$v_{r,m,s} = 0.201 m.s^{-1}$, (b) for 628 pedestrians on a pedestrian crossing in Sydney, $v_x = 1.44 m.s^{-1}$,$v_{r,m,s} = 0.228 m.s^{-1}$

According to the above and the last remark, we define in our case the density of equilibrium for pedestrians, with the following formulation:

$$f_{eq}(\mathbf{v},\rho) = \frac{\rho(t,\mathbf{x})}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}-\mathbf{v}_d||^2}{v_m^2}\Big), \tag{2.8}$$

where, $v_m = \sqrt{\frac{1}{2}\int_{D_\mathbf{v}} ||\mathbf{v}||^2 f(t,\mathbf{x},\mathbf{v})d\mathbf{v}}$; $\mathbf{v}_d$ is the desired direction.
f_{eq} models the equilibrium state that each pedestrian wishes to achieve. Indeed the tendency of all pedestrians to reach a comfortable destination defined by the direction $\mathbf{v}_d$.
The density of equilibrium that we have defined satisfies the conservation of the mass defined by the equation (2.5), indeed:

$$\int_{D_\mathbf{v}} f_{eq}(\mathbf{v})d\mathbf{v} = \int_{D_\mathbf{v}} \frac{\rho(t,\mathbf{x})}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}-\mathbf{v}_d||^2}{v_m^2}\Big)d\mathbf{v} = \rho(t,\mathbf{x}) = \int_{D_\mathbf{v}} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}.$$

2.3. **Model of discrete velocities.** The number of pedestrians N is generally insufficient to justify the hypothesis of continuity of the particle distribution function $f(t,\mathbf{x},\mathbf{v})$ with respect to velocity. Thus for numerical simulations, a discrete velocity approximation of the BGK equation is introduced. We refer to the discrete velocity models for the Boltzmann equation developed by Rogier and Schneider [22], Buet [23], Heintz and Panferov [24], and Mieussens [25]. The proposed approximation in this work has the same conservation properties as the continuous BGK model.
Let $\mathcal{K}$ be a set of multi-indices of $\mathbb{Z}^2$, defined by $\mathcal{K} = \{\mathbf{k} = (k_1,k_2), |\mathbf{k}| \leq B\}$, with B is a scalar.
We define $\mathcal{V} \subset \mathbb{R}^2$, a set of N_v discrete velocities, $\mathcal{V} = \{\mathbf{v}_\mathbf{k} = \mathbf{k}\Delta v \ /\mathbf{k} \in \mathcal{K}\}$, where Δv is a scalar.
The distribution f of the continuous velocities is then replaced by the following N_v vector, $f_\mathcal{K} = (f_\mathbf{k}(t,\mathbf{x}))_{\mathbf{k}\in\mathcal{K}}$ where each component $f_\mathbf{k}(t,\mathbf{x})$ is an approximation of the function $f(t,\mathbf{x},\mathbf{v}_\mathbf{k})$.
Thanks to the previous discretization of the velocity, we define the local density by the following equation:

$$\rho\,(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f_\mathbf{k}\,(\mathbf{x},t)\,. \tag{2.9}$$

The discrete kinetic model associated with the equation BGK (2.2), is defined by the set of the following equations:

$$\frac{\partial f_\mathbf{k}(t,\mathbf{x})}{\partial t} + \mathbf{v}_\mathbf{k}.\nabla_\mathbf{x} f_\mathbf{k}(t,\mathbf{x}) = \frac{1}{\tau}\left(f_{eq,\mathbf{k}}\left(\mathbf{v}_\mathbf{k},\rho\right) - f_\mathbf{k}(t,\mathbf{x})\right), \quad \mathbf{k}\in\mathcal{K} \ \ \mathbf{v}_\mathbf{k}\in\mathcal{V}, \tag{2.10}$$

where $f_{eq,\mathbf{k}}$ is an approximation of the equilibrium density defined by (2.8) .
The main problem is to define this approximation of the discrete density $f_{eq,\mathbf{k}}$ such that the property of conservation of mass is satisfied. We used the natural approximation used by Yang and Huang [26] and which has been developed by Luc [27]:

$$f_{eq,\mathbf{k}}\left(\mathbf{v}_\mathbf{k},\rho\right) = f_{eq}(\mathbf{v}_\mathbf{k}),\ \ \mathbf{k}\in\mathcal{K}, \tag{2.11}$$

hence,

$$f_{eq}(\mathbf{v}_\mathbf{k},\rho) = \frac{\rho(t,\mathbf{x})}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}_\mathbf{k}-\mathbf{v}_d||^2}{v_m^2}\Big), \tag{2.12}$$

where

$$v_m = \sqrt{\frac{1}{\rho(t,\mathbf{x})}\sum_{\mathbf{k}\in\mathcal{K}} ||\mathbf{v}_\mathbf{k}||^2 f_\mathbf{k}(t,\mathbf{x})}.$$

We considered the mathematical model (2.10) with an initial data defined by:

$$f_\mathbf{k}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) \quad \mathbf{k}\in\mathcal{K}.\ \mathbf{x}\in\Omega. \tag{2.13}$$

Boundary condition on $\partial\Omega$
In our study, we are interested in the adaptation of an equilibrium situation of the pedestrians inside the domain Ω. We assume that their target is inside the domain. The disk diameter $2r_D$, occupied by pedestrians is always less than the distance between the target in $\mathbf{x}_c$ and the edge of the domain $\partial\Omega$, i.e.

$$2r_D < d\left(\mathbf{x}_c, \partial\Omega\right).$$

According to this hypothesis, the theoretical study of our problem is reduced to all plane $\mathbb{R}^2$ in places Ω.

3. Theoretical study of the proposed mathematical model

Some important mathematical results concerning the BGK equation have been obtained during the last decade. For example, Perthame has proved in [28] the existence and stability of a distribution solution throughout the space. This result has been extended to a bounded domain with various boundary conditions by Ringeisen [29]. More recently, Perthame and Pulvirenti have proved the existence and uniqueness of a "mild" solution with weighted estimates in $\mathbb{L}^\infty$ [30]. We also mention the result of Issautier [31] which proved that the "mild" solution of Perthame and Pulvirenti is strong, if certain assumptions of regularity on the initial condition are made. However, it is important to note that in all these results, the authors assume the relaxation time is constant (i.e. $\tau = 1$).
In our study we are interested in the existence and uniqueness of the BGK model with a different source or a different equilibrium density to that defined in the case of fluid dynamics. Namely a density that is suitable for pedestrian movement. In addition, we assume that $\tau = 1$.
It is interesting to study the convergence of such an approximation to the continuous BGK equation. We refer to Mischler's proof for the convergence of a discrete velocity model for the Boltzmann equation [32]. There are essentially two distinct points to prove:

- The existence and uniqueness of a discrete velocity model solution.
- Convergence of the discrete kinetic equation towards the continuous equation.

3.1. Existence and uniqueness of the model solution.

To define a discrete velocity model of approximation 2.10, we consider the following notations:
We consider $\mathcal{V}^n$ a grid of N_n velocities defined by: $\mathcal{V}^n = \{\mathbf{v}^n_{\mathbf{k}} = \mathbf{k}\Delta v_n \ /\mathbf{k} \in \mathcal{K}^n\}$.
where, $\mathcal{K}^n = \left\{\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2, |\mathbf{k}| \leq B_n\right\}$.
Δv_n, B_n are two real suites, assumed such that:

$$\Delta v_n \underset{n \to +\infty}{\longrightarrow} 0 \qquad\qquad \Delta v_n B_n \underset{n \to +\infty}{\longrightarrow} +\infty.$$

We also define velocity's cells $\mathcal{I}^n_{\mathbf{k}}$ par $\mathcal{I}^n_{\mathbf{k}} = [v^n_{k_1}, v^n_{k_1} + \frac{1}{2}\Delta v_n[\times[v^n_{k_2}, v^n_{k_2} + \frac{1}{2}\Delta v_n[$.
The discrete velocity model approximation 4.1 is then given by the following systems:

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial f^n_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \mathbf{v}^n_{\mathbf{k}}.\nabla_{\mathbf{x}} f^n_{\mathbf{k}}(t,\mathbf{x}) = f^n_{eq,\mathbf{k}}\left(\mathbf{v}_{\mathbf{k}}, \rho\right) - f^n_{\mathbf{k}}\left(t,\mathbf{x}\right)) \quad \mathcal{D}'\left([0,T]\times\mathbb{R}^2_{\mathbf{x}}\right) \ \mathbf{k} \in \mathcal{K}, \\ f^n_{\mathbf{k}}(t=0,\mathbf{x}) = f^n_{0,\mathbf{k}}(\mathbf{x}) \qquad \mathbf{k} \in \mathcal{K}, \end{cases}$$

where $f^n_{0,\mathbf{k}}$ is an approximation of the initial density f_0,

$$\text{(3.2)} \qquad \mathcal{D}'\left([0,T]\times\mathbb{R}^2_{\mathbf{x}}\right) = \Big\{W : \mathcal{D}\left([0,T]\times\mathbb{R}^2_{\mathbf{x}}\right) \longrightarrow \mathbb{R}, \quad W \text{ continuous, linear}\Big\},$$

$$\text{(3.3)} \qquad \mathcal{D}\left([0,T]\times\mathbb{R}^2_{\mathbf{x}}\right) = \Big\{f \in \mathcal{C}^\infty\left([0,T]\times\mathbb{R}^2_{\mathbf{x}}\right) : \quad \text{supp(f), compact}\Big\}.$$

Our goal is to show the existence and uniqueness of the model 3.1.
We consider the characteristic curves associated with the problem 3.1, that are given by:

$$\boldsymbol{\gamma}_{\mathbf{k}}(t) = \left(\gamma_{k_1}(t), \gamma_{k_2}(t)\right) = \left(x + tv_{k_1}^n, y + tv_{k_2}^n\right), \ \mathbf{k} = (k_1, k_2) \in \mathcal{K}. \tag{3.4}$$

These curves are solutions of the following equations:

$$\begin{cases} \dfrac{d\boldsymbol{\gamma}_{\mathbf{k}}(t)}{dt} = \mathbf{v}_{\mathbf{k}}^n \ \mathbf{k} \in \mathcal{K}, \\ \boldsymbol{\gamma}_{\mathbf{k}}(0) = (x, y)^T \ \ \mathbf{k} \in \mathcal{K}. \end{cases} \tag{3.5}$$

with $\mathbf{v}_{\mathbf{k}}^n = \left(v_{k_1}^n, v_{k_2}^n\right)$.
Along these curves, the system solution 3.1 satisfies the following system of ordinary differential equations:

$$\begin{cases} \dfrac{df_{\mathbf{k}}^n(t, \boldsymbol{\gamma}_{\mathbf{k}}(t))}{dt} = f_{eq,\mathbf{k}}^n\left(\mathbf{v}_{\mathbf{k}}, \rho\right) - f_{\mathbf{k}}^n\left(t, \boldsymbol{\gamma}_{\mathbf{k}}(t)\right) \qquad \mathbf{k} \in \mathcal{K}, \\ f_{\mathbf{k}}^n(t = 0, \mathbf{x}) = f_{0,\mathbf{k}}^n(\mathbf{x}) \qquad \mathbf{k} \in \mathcal{K}. \end{cases} \tag{3.6}$$

We pose $\widehat{f_{\mathbf{k}}^n}\left(t, \mathbf{x}\right) = f_{\mathbf{k}}^n(t, \boldsymbol{\gamma}_{\mathbf{k}}(t))$ for $\mathbf{k} \in \mathcal{K}$, where, $\widehat{f_{\mathbf{k}}^n}$ is the value of f^n along these characteristic curves.
We introduce the "mild" form of the system (3.6) obtained by integration along the characteristic curves (3.4), for $\mathbf{k} \in \mathcal{K}$,

$$\widehat{f_{\mathbf{k}}^n}\left(t, \mathbf{x}\right) = f_{0,\mathbf{k}}^n + \int_0^t \left(\widehat{f_{eq,\mathbf{k}}^n}\left(\mathbf{v}_{\mathbf{k}}, \rho\right) - \widehat{f_{\mathbf{k}}^n}\left(s, \mathbf{x}\right)\right) ds\,. \tag{3.7}$$

For a given time, we define the following functional space:

$$\mathbb{L}^1\left(\mathbb{R}_{\mathbf{x}}^2\right) = \left\{ \mathbf{f}(t) = \left(f_{\mathbf{k}}\right)_{\mathbf{k}\in\mathcal{K}}, \ \ ||\mathbf{f}(t)||_1 = \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}_{\mathbf{x}}^2} ||f_{\mathbf{k}}(t, \mathbf{x})|| d\mathbf{x} < \infty \right\}.$$

For a time $T > 0$, consider the following Banach space:
$\mathbb{X}_T = \mathcal{C}\left([0, T], \mathbb{L}^1\left(\mathbb{R}_{\mathbf{x}}^2\right)\right)$, with the following norm: $||\mathbf{f}||_{\mathbb{X}_T} = sup_{t\in[0,T]}||\mathbf{f}(t)||_1$.
Our theoretical study is based on the following theorem:

Theorem 1. *(Local existence)*
Let $f_0^n = \left(f_{0,\mathbf{k}}^n\right)_{\mathbf{k}\in\mathcal{K}} \in \mathbb{L}^\infty\left(\mathbb{R}_{\mathbf{x}}^2\right) \cap \mathbb{L}^1\left(\mathbb{R}_{\mathbf{x}}^2\right)$ *with* $f_0^n \geq 0$, *then there is a time* $T > 0$ *and a constant* R *such as* $\forall t < T$, *the problem 3.1 admits a unique solution* $f^n = \left(f_{\mathbf{k}}^n\right)_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}\left([0, T], \mathbb{L}^1\left(\mathbb{R}_{\mathbf{x}}^2\right)\right)$ *and which satisfies the following estimates:*

$$sup_n sup_{[0,T]} \int_{\mathbb{R}_{\mathbf{x}}^2} \sum_{\mathbf{k}\in\mathcal{K}} f_{\mathbf{k}}^n\left(t, \mathbf{x}\right) d\mathbf{x} \leq \Theta\left(T\right), \tag{3.8}$$

and

$$\rho(t, \mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f_{\mathbf{k}}^n\left(t, \mathbf{x}\right) \leq R. \tag{3.9}$$

If furthermore $\sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}^n||_\infty < 1$ *then the solution of the model has a physical meaning i.e. :*

$$\rho(t, \mathbf{x}) \leq 1. \tag{3.10}$$

Since this theorem is independent of n, for simplicity, the exponent n is omitted in this section.

Proof of the theorem

We introduce the following function:

$$\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) = \widehat{f}_{\mathbf{k}}\,(t,\mathbf{x})\, exp\,(\lambda t) \quad for \ \ \mathbf{k}\in\mathcal{K} \ \ \lambda > 0.$$

Therefore, the system 3.6 equivalent to the following system:

$$\begin{cases} \dfrac{d\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x})}{dt} = \lambda\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}\,(t,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) \qquad \mathbf{k}\in\mathcal{K}, \\ \widehat{\psi}_{\mathbf{k}}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) \qquad \mathbf{k}\in\mathcal{K}. \end{cases} \tag{3.11}$$

For all $t\in[0,T]$, we integrate the equation 3.11, we deduce the following "mild" formulation:

$$\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \Big(\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}\,(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\Big)\,ds \ \ \forall\,\mathbf{k}\in\mathcal{K}.$$

Consider the following operator $\mathbf{A} = \left(\widehat{\mathbf{A}\psi_{\mathbf{k}}}\right)_{\mathbf{k}\in\mathcal{K}}$:

$$\widehat{\mathbf{A}\psi_{\mathbf{k}}}(t,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \Big(\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}\,(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\Big)\,ds \ \ \forall\,\mathbf{k}\in\mathcal{K}.$$

To show that the system (3.11) has a solution, it is enough to show that the operator $\boldsymbol{A}$ has a unique fixed point in the Banach space $\mathbb{X}_T$. Indeed, we introduce the set defined by:

$$B_{T,a_0,\lambda,R} = \Bigg\{\widehat{\boldsymbol{\psi}} = \left(\widehat{\psi}\right)_{\mathbf{k}\in\mathcal{K}} \in \mathbb{X}_T: \quad \widehat{\psi}_{\mathbf{k}} \geq 0, \quad ||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} \leq a_0||f_0||_1, \\ \sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}\,(t,\mathbf{x}) \leq R\, exp(\lambda t), \ \ t\in[0,T], \quad \mathbf{x}\in\mathbb{R}^2_{\mathbf{x}}\Bigg\}.$$

Lemma 1. *Let* $\widehat{\boldsymbol{\psi}} \in B_{T,a_0,\lambda,R}$,

(1) There is λ_0*, such that* $\forall\lambda\geq\lambda_0$*, we have :* $\left(\widehat{\boldsymbol{A\psi}}\right)_{\mathbf{k}\in\mathcal{K}} \geq 0.$

(2) If $\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}\,(t,\mathbf{x}) \leq R\,exp(\lambda t)$ *then, there are two constant* R_0, T*, such that:* $\forall\, R\geq R_0 \, and \, t\in[0,T]$,

$$\sum_{\mathbf{k}\in\mathcal{K}} \widehat{A\psi_{\mathbf{k}}}\,(t,\mathbf{x}) \leq R\,exp(\lambda t).$$

(3) Let $C_2 = \left(\frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\right) - 1\right)$ *such that* $(C_2+\lambda)\,T \neq 1$*, then there is a constant* a_0 *such that:*

$$||\widehat{\boldsymbol{A\psi}}||_{\mathbb{X}_T} \leq a_0||f_0||_1 \tag{3.12}$$

(4) Let $\widehat{\boldsymbol{\psi}^{\mathbf{1}}}, \widehat{\boldsymbol{\psi}^{\mathbf{2}}} \in B_{T,a_0,\lambda,R}$*, then there exist* C_3 *such that:*

$$||\widehat{\boldsymbol{A\psi}^{\mathbf{1}}}(t) - \widehat{\boldsymbol{A\psi}^{\mathbf{2}}}(t)||_1 \leq (\lambda T + C_3 T + T)\,||\widehat{\boldsymbol{\psi}^{\mathbf{1}}} - \widehat{\boldsymbol{\psi}^{\mathbf{2}}}||_{\mathbb{X}_T}$$

Proof of the lemma:

(1) Since $f_{0,\mathbf{k}}(\mathbf{x}) \geq 0$ $\mathbf{k} \in \mathcal{K}$, then $\left(\widehat{\boldsymbol{A\psi}}\right)_{\mathbf{k}\in\mathcal{K}} \geq 0$ if,

$$\left(\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\right) \geq 0 \ \mathbf{k} \in \mathcal{K}.$$

We have

$$\begin{aligned}\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) &= \lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \frac{\sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_k(s,\mathbf{x})}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||v-\mathbf{v}_d||^2}{v_m^2}\Big) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \\ &= \left(\lambda + \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\Big) - 1\right)\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \frac{\sum_{\mathbf{l}\in\mathcal{K}\ \mathbf{l}\neq\mathbf{k}}\widehat{\psi}_{\mathbf{l}}(s,\mathbf{x})}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||v-\mathbf{v}_d||^2}{v_m^2}\right).\end{aligned}$$

Since $\frac{\sum_{\mathbf{l}\in\mathcal{K}\ \mathbf{l}\neq\mathbf{k}}\widehat{\psi}_{\mathbf{l}}(s,\mathbf{x})}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||v-\mathbf{v}_d||^2}{v_m^2}\right) \geq 0$ then $\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \geq 0$ if:

$$\left(\lambda + \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\Big) - 1\right) \geq 0,$$

hence, if:

$$\lambda \geq \lambda_0 = 1 - \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\Big).$$

(2) we have $\sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) \leq R\,exp(\lambda t)$, hence:

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{K}}\widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x}) &= \sum_{\mathbf{k}\in\mathcal{K}} f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \left(\sum_{\mathbf{k}\in\mathcal{K}}\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_{eq,\mathbf{k}}(s,\mathbf{x}) - \sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\right) ds \\ &= \sum_{\mathbf{k}\in\mathcal{K}} f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\left(\sum_{\mathbf{k}\in\mathcal{K}}\frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\right) - 1\right) + \sum_{\mathbf{k}\in\mathcal{K}}\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})ds. \\ &\leq \sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty + \frac{C_1 R}{\lambda}(exp(\lambda t) - 1) + R\,exp(\lambda t) - R.\end{aligned}$$

with $C_1 = \left(\sum_{\mathbf{k}\in\mathcal{K}}\frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\right) - 1\right)$.
Assuming that the choice of $\mathbf{v}_{\mathbf{k}}$ and $\mathbf{v}_d$, is made such that $C_1 > 0$.
Hence $\sum_{\mathbf{k}\in\mathcal{K}}\widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x}) \leq R\,exp(\lambda t)$ if,

$$\sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty + \frac{C_1 R}{\lambda}(exp(\lambda t) - 1) - R \leq 0 \tag{3.13}$$

$$\text{hence } R \geq R_1 = \sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty \tag{3.14}$$

$$\text{and, } t \leq T = \frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{C_1 R}\left(R - \sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty\right)\right) \tag{3.15}$$

(3) Since $\widehat{\boldsymbol{\psi}} \in B_{T,a_0,\lambda,R}$, we have $||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} \leq a_0||f_0||_1$, moreover,

$$\sum_{\mathbf{k}\in\mathcal{K}}\int_{\mathbb{R}^2_{\mathbf{x}}} |\widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x})|d\mathbf{x} = \sum_{\mathbf{k}\in\mathcal{K}}\int_{\mathbb{R}^2_{\mathbf{x}}} |f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \left(\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\right) ds|d\mathbf{x} \tag{3.16}$$

$$\leq ||f_0||_1 + \lambda T||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} + \int_{\mathbb{R}^2_{\mathbf{x}}} |\int_0^t \sum_{\mathbf{k}\in\mathcal{K}}\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})\left(\frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d||^2}{v_m^2}\right) - 1\right) ds|d\mathbf{x} \tag{3.17}$$

$$\leq ||f_0||_1 + \lambda T||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} + TC_2||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} \tag{3.18}$$

$$\leq (1 + \lambda a_0 T + T a_0 C_2)\,||f_0||_1, \tag{3.19}$$

with $C_2 = \left(\frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\right) - 1\right)$.

Whence, $||\widehat{\boldsymbol{A\psi}}||_{\mathbb{X}_T} \leq a_0||f_0||_1$,
if $(C_2+\lambda)\,T \neq 1$, i.e. $\lambda \neq \frac{1}{T} - C_2$ and a_0, satisfies the following equation: $1 + \lambda a_0 T + T a_0 C_2 = a_0$. Indeed, the constant a_0 is given by:

$$a_0 = \frac{1}{1-(C_2+\lambda)\,T} \quad \text{where }, \; (C_2+\lambda)\,T \neq 1. \tag{3.20}$$

(4)

$$\begin{aligned}
||\widehat{\boldsymbol{A\psi^1}}(t) - \widehat{\boldsymbol{A\psi^2}}(t)||_1 &= \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} |\widehat{A\psi^1}_{\mathbf{k}}(t,\mathbf{x}) - \widehat{A\psi^2}_{\mathbf{k}}(t,\mathbf{x})\,| d\mathbf{x} \\
&= \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} |\int_0^t \lambda\left(\widehat{\psi}^1_{\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}^2_{\mathbf{k}}(s,\mathbf{x})\right) ds + \int_0^t \left(\widehat{\psi}^1_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}^2_{eq,\mathbf{k}}(s,\mathbf{x})\right) ds \\
&+ \int_0^t \left(\widehat{\psi}^2_{\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}^1_{\mathbf{k}}(s,\mathbf{x})\right) ds | d\mathbf{x}.
\end{aligned}$$

By definition of the equilibrium density, we have,

$$\widehat{\psi}^2_{eq,\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}^1_{eq,\mathbf{k}}(s,\mathbf{x}) = \frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\right)\left(\sum_{\mathbf{l}\in\mathcal{K}} \widehat{\psi}^1_{\mathbf{l}}(s,\mathbf{x}) - \sum_{\mathbf{l}\in\mathcal{K}} \widehat{\psi}^2_{\mathbf{l}}(s,\mathbf{x})\right)$$

We pose $C_3 = \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} exp\left(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\right)$
Hence by a change of variable, we find:

$$\begin{aligned}
||\widehat{\boldsymbol{A\psi^1}}(t) - \widehat{\boldsymbol{A\psi^2}}(t)||_1 &\leq \lambda T||\widehat{\boldsymbol{\psi^1}} - \widehat{\boldsymbol{\psi^2}}||_{\mathbb{X}_T} + C_3 T||\widehat{\boldsymbol{\psi^1}} - \widehat{\boldsymbol{\psi^2}}||_{\mathbb{X}_T} + T||\widehat{\boldsymbol{\psi^1}} - \widehat{\boldsymbol{\psi^2}}||_{\mathbb{X}_T} \\
&\leq (\lambda T + C_3 T + T)\,||\widehat{\boldsymbol{\psi^1}} - \widehat{\boldsymbol{\psi^2}}||_{\mathbb{X}_T}.
\end{aligned}$$

Which ends the proof of the lemma.
According to 1, 2, 3 of the lemma, for any $\lambda_0 \leq \lambda$ $(\lambda \neq \frac{1}{T} - C_2)$, $R \geq R_1$, $t \leq T$ and a_0 verifies (3.20), we have: if $\widehat{\boldsymbol{\psi}} \in B_{T,a_0,\lambda,R}$ then $\widehat{\boldsymbol{A\psi}} \in B_{T,a_0,\lambda,R}$.
According to 4 of the lemma, we have

$$||\widehat{\boldsymbol{A\psi^1}} - \widehat{\boldsymbol{A\psi^2}}||_{\mathbb{X}_T} \leq (\lambda T + C_3 T + T)\,||\widehat{\boldsymbol{\psi^1}} - \widehat{\boldsymbol{\psi^2}}||_{\mathbb{X}_T}.$$

Moreover $\lambda < \left(\frac{1}{T} - (C_3+1)\right) := \lambda_1$ then:

$$(\lambda T + C_3 T + T) < 1.$$

Hence the operator $\mathbf{A} : B_{T,a_0,\lambda,R} \to B_{T,a_0,\lambda,R}$ is a contraction.
Banach's fixed point theorem refers to the local existence of the model solution.
From the foregoing, there exist λ, $(\lambda_0 \leq \lambda < \lambda_1,\ \lambda \neq \frac{1}{T} - C_2)$, T, a_0 and R, such that the problem 3.1 has a unique positive solution $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}\left([0,T], \mathbb{L}^1\left(\mathbb{R}^2_{\mathbf{x}}\right)\right)$ and which satisfies :

$$\int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\, d\mathbf{x} \leq a_0||f_0||_1\, exp(-\lambda t) \quad (\text{ since } ||\widehat{\boldsymbol{\psi}}||_{\mathbb{X}_T} \leq a_0||f_0||_1).$$

Hence,

$$sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\, d\mathbf{x} \leq a_0||f_0||_1.$$

Since $a_0||f_0||_1$ does not depend on n and $a_0 = a_0(T)$, then:

$$sup_n sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\, d\mathbf{x} \leq \Theta(T).$$

where, $\Theta(T) = a_0(T)||f_0||_1$, from where the estimation (3.8).
In addition the solution satisfies

$$\rho(t,\mathbf{x}) \leq R, \quad t \quad \in [0,T] \quad \mathbf{x} \in \Omega, \quad R \geq R_1,$$

where

$$R_1 = \sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty.$$

Moreover if $\sum_{\mathbf{k}\in\mathcal{K}} ||f_{0,\mathbf{k}}||_\infty < 1$, $(R_1 \leq 1)$, then we choose $R = R_1$ such that

$$\rho(t,\mathbf{x}) \leq 1, \quad \forall \quad t \quad \in [0,T], \quad \forall \quad \mathbf{x} \in \Omega,$$

Hence the estimation (3.10). That ends the proof of the theorem 1.

Remark 2. *In our proof we have demonstrated the local existence of problem 3.1, in [0,T]. Through an iteration process. We can successively solve the equation 3.1 with initial conditions in $t_0 = 0$ until T, $t_1 = T$ until $t_2 = 2T$, ...*
By concatenation we build a maximum solution on $[0, T_{max}[$, with $T_{max} = \sup_j t_j$. This solution belongs $\mathcal{C}\left([0,T_{max}[, \mathbb{L}^1\left(\mathbb{R}^2_{\mathbf{x}}\right)\right)$

3.2. **Convergence of the discrete kinetic equation towards the continuous one.** According to the previews section, we have shown the existence and uniqueness of the discrete model solution(3.1) $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}\left([0,T_{max}[, \mathbb{L}^1\left(\mathbb{R}^2_{\mathbf{x}}\right)\right)$, moreover satisfies the following estimate:

$$sup_n sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\, d\mathbf{x} \leq \Theta(T). \tag{3.21}$$

In order to prove the convergence of this solution, we define the following functions:

$$f^n(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x}) \mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.22}$$

$$f^n_{eq}(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{eq,\mathbf{k}}(t,\mathbf{x}) \mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.23}$$

$$C^n(\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} \mathbf{v}^n_{\mathbf{k}} \mathbb{1}^n_{\mathbf{k}}(\mathbf{v}) \tag{3.24}$$

$$f^0(0,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{0,n}_{\mathbf{k}}(\mathbf{x}) \mathbb{1}^n_{\mathbf{k}}(\mathbf{v}) \tag{3.25}$$

with $\mathbb{1}^n_{\mathbf{k}}$ the indicator function on the velocity cells $\mathcal{I}^n_{\mathbf{k}}$.
Then the discrete model 3.1 can be linked to the BGK equation 2.1 by the following equation:

$$\begin{cases} \dfrac{\partial f^n(t,\mathbf{x},\mathbf{v})}{\partial t} + C^n(\mathbf{v}).\nabla_{\mathbf{x}} f^n(t,\mathbf{x},\mathbf{v}) = f^n_{eq}(\mathbf{v}) - f^n(t,\mathbf{x},\mathbf{v}) \quad \mathcal{D}'\left(]0,T_{max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}}\right), \\ f^0(0,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{0,n}_{\mathbf{k}}(\mathbf{x}) \mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \end{cases} \tag{3.26}$$

where,

$$\mathcal{D}^{'}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right) = \Big\{W : \mathcal{D}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right) \longrightarrow \mathbb{R}, \quad W \,\text{continuous, linear}\Big\}, \tag{3.27}$$

$$\mathcal{D}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right) = \Big\{f\in\mathcal{C}^{\infty}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right) : \quad \text{supp(f), compact}\Big\}. \tag{3.28}$$

We now denote our convergence result in the following theorem:

Theorem 2. *let* $\mathbf{v}^n$ *such that* $\Delta v_n \underset{n\to+\infty}{\longrightarrow} 0 \qquad \Delta v_n B_n \underset{n\to+\infty}{\longrightarrow} +\infty$, *and*
$\begin{array}{cccc} C^n & : & \mathbb{R}^2_{\mathbf{v}} & \to & \mathbb{R}^2_{\mathbf{v}} \\ & & \mathbf{v} & \mapsto & C^n(\mathbf{v}) \end{array}$ *such that:*

$$C^n \text{ is locally, uniformly bounded in } \left(\mathbb{L}^{\infty}_{loc}\left(\mathbb{R}^2_{\mathbf{v}}\right)\right)^2, \tag{3.29}$$

$$C^n(\mathbf{v}) \underset{n\to+\infty}{\longrightarrow} \mathbf{v} \qquad \text{simply.} \tag{3.30}$$

Then we can extract a sub-sequence noted $(f^n)_n$ *which converges weakly in* $\mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, $\forall T_{max}\geq 0$ *to a solution of the equation* (2.1)*.*

Proof
We refer to the works of Perthame [28] and [25], we divide the proof into 4 steps:
step 1: weak convergence of f^n
According to (3.21), (3.22), it is clear that f^n satisfies the following estimate:

$$sup_n sup_{[0,T]}\int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(t,\mathbf{x},\mathbf{v}\right)d\mathbf{x}d\mathbf{v} \leq \Theta_1\left(T\right), \tag{3.31}$$

which implies that the following $(f^n)_n$ is equi-integrable.

According to Dunford-Pettis theorem [33], we can extract a subsequence noted $(f^n)_n$ which converges weakly in $\mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, towards $f\in\mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, i.e.:

$$f^n \underset{n\to+\infty}{\rightharpoonup} f \qquad \text{in } \mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right). \tag{3.32}$$

Hence

$$f^n \underset{n\to+\infty}{\longrightarrow} f \qquad \text{in } \mathcal{D}^{'}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right),$$

for all $T_{max}\geq 0$. We thus obtain the convergence of the transport term of (3.26) towards $\partial f + \mathbf{v}.\nabla f$ in $\mathcal{D}^{'}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$., i.e.:

$$\partial f^n + C^n(\mathbf{v}).\nabla_{\mathbf{x}} f^n \underset{n\to+\infty}{\longrightarrow} \partial f + \mathbf{v}.\nabla_{\mathbf{x}} f \qquad \text{in } \mathcal{D}^{'}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right). \tag{3.33}$$

Indeed, let $\varphi\in\mathcal{D}\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$,

$$\begin{aligned}\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}}\left(\partial_t f^n + C^n(\mathbf{v}).\nabla_{\mathbf{x}} f^n\right)\varphi dt d\mathbf{x} d\mathbf{v} &= -\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(\partial_t\varphi + C^n(\mathbf{v}).\nabla_{\mathbf{x}}\varphi\right)dt d\mathbf{x} d\mathbf{v}\\ &= -\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(\partial_t\varphi\right)dt d\mathbf{x} d\mathbf{v} - \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(C^n(\mathbf{v}).\nabla_{\mathbf{x}}\varphi\right)dt d\mathbf{x} d\mathbf{v}.\end{aligned}$$

We have :

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(C^n(\mathbf{v}).\nabla_{\mathbf{x}}\varphi\right)dtd\mathbf{x}d\mathbf{v} = \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(C^n(\mathbf{v})-\mathbf{v}\right).\nabla_{\mathbf{x}}\varphi dtd\mathbf{x}d\mathbf{v} + \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\mathbf{v}.\nabla_{\mathbf{x}}\varphi dtd\mathbf{x}d\mathbf{v}$$

According to (3.29), (3.30)

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(C^n(\mathbf{v})-\mathbf{v}\right).\nabla_{\mathbf{x}}\varphi dtd\mathbf{x}d\mathbf{v} \le ||f^n||_{\mathbb{L}^1(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})}||C^n(\mathbf{v})-\mathbf{v}||_{\mathbb{L}^\infty(\mathbb{R}^2_{\mathbf{v}})}||\nabla_{\mathbf{x}}\varphi||_{\mathbb{L}^\infty(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})}.$$

From (3.30), $||C^n(\mathbf{v})-\mathbf{v}||_{\mathbb{L}^\infty(\mathbb{R}^2_{\mathbf{v}})} \underset{n\to+\infty}{\longrightarrow} 0$, consequently,

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(C^n(\mathbf{v})-\mathbf{v}\right).\nabla_{\mathbf{x}}\varphi dtd\mathbf{x}d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.34}$$

Moreover, since $\mathbf{v}.\nabla_{\mathbf{x}}\varphi \in \mathbb{L}^\infty\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, then

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\mathbf{v}.\nabla_{\mathbf{x}}\varphi dtd\mathbf{x}d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f\mathbf{v}.\nabla_{\mathbf{x}}\varphi dt. \tag{3.35}$$

In addition, $\partial_t\varphi \in \mathbb{L}^\infty\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, then

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(\partial_t\varphi\right)dtd\mathbf{x}d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f\left(\partial_t\varphi\right)dtd\mathbf{x}d\mathbf{v}. \tag{3.36}$$

From, (3.34),(3.35) and (3.36), we have

$$\int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n\left(\partial_t\varphi + C^n(\mathbf{v}).\nabla_{\mathbf{x}}\varphi\right)dtd\mathbf{x}d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \int_{]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f\left(\partial_t\varphi + \mathbf{v}.\nabla_{\mathbf{x}}\varphi\right)dtd\mathbf{x}d\mathbf{v},$$

hence the result (3.33).
For the convergence of the non-linear part, we first have the convergence of ρ^n, according to (3.32):

$$\int_{\mathbb{R}^2_{\mathbf{v}}} f^n(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} \qquad \text{weakly in } \mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\right),$$

i.e.

$$\rho^n(t,\mathbf{x}) \underset{n\to+\infty}{\rightharpoonup} \rho(t,\mathbf{x}) \qquad \text{in } \mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\right). \tag{3.37}$$

Step 2 : weak convergence of f^n_{eq}
We need the following lemma:

Lemma 2. *Suppose that $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}}$ satisfies the inequality (3.21), then $\forall T \ge 0$, there is $C(T)$ such that*

$$sup_n sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{eq}(t,\mathbf{x},\mathbf{v})\,d\mathbf{x}d\mathbf{v} \le C(T), \tag{3.38}$$

Proof of the lemma 2
We have

$$f^n_{eq}(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{eq,\mathbf{k}}(t,\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}),$$

with, $f^n_{eq,\mathbf{k}}(t,\mathbf{x}) = \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\Big)$, $\rho\left(t,\mathbf{x}\right) = \sum_{\mathbf{k}\in\mathcal{K}} f_{\mathbf{k}}\left(\mathbf{x},t\right)$.

$$\begin{aligned}\int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{eq}\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{x}d\mathbf{v} &= \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}\left(\mathbf{x},t\right) \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\Big)\mathbb{1}^n_{\mathbf{k}}(\mathbf{v})d\mathbf{x}d\mathbf{v} \\ &= \int_{\mathbb{R}^2_{\mathbf{x}}\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}\left(\mathbf{x},t\right) d\mathbf{x} \int_{\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\Big)\mathbb{1}^n_{\mathbf{k}}(\mathbf{v})d\mathbf{v} \\ &= \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}\left(\mathbf{x},t\right) d\mathbf{x} \int_{\mathcal{I}^n_{\mathbf{k}}} \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} exp\Big(-\frac{1}{2}\frac{||\mathbf{v_k}-\mathbf{v}_d||^2}{v_m^2}\Big)d\mathbf{v}\end{aligned}$$

from where,

$$\begin{aligned}sup_n sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{eq}\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{x}d\mathbf{v} &\leq sup_n sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}\left(\mathbf{x},t\right) d\mathbf{x}.C\left(\mathcal{I}^n_{\mathbf{k}}\right) \\ &\text{according to (3.21)} \leq C\left(T\right),\end{aligned}$$

where $C\left(T\right) := \Theta(T).C\left(\mathcal{I}^n_{\mathbf{k}}\right)$.

According to this lemma (f^n_{eq}) is weakly compact in $\mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, so we can extract a subsequence noted (f^n_{eq}) such that $f^n_{eq} \underset{n\to+\infty}{\rightharpoonup} g \quad \text{in}\, \mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right)$, where from the step (1) and (2), the low limit of f^n satisfies the equation

$$\partial f + \mathbf{v}.\nabla f = g - f \qquad \text{in } \mathcal{D}'\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right). \tag{3.39}$$

The next step is to show that $g = f_{eq}$

Step 3: strong convergence of ρ^n.

According to the compactness lemma on averages obtained by Mischler [32], and [?], on sets of bounded velocities f^n is strongly compact therefore. Indeed, we have for any R_C (extracting again subsequences):

$$\int_{|\mathbf{v}|\leq R_C} f^n\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \int_{|\mathbf{v}|\leq R_C} f\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{v}, \tag{3.40}$$

strongly in $\mathbb{L}^1\left(]0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}\right)$.

From the uniform estimates (3.31), we thus obtain

$$f^n \underset{n\to+\infty}{\longrightarrow} f \qquad \text{in } \mathbb{L}^1\left(]0,T_{max}[\times K_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}\right). \qquad \forall\, K_{\mathbf{x}} \text{ compact of}\, \mathbb{R}^2_{\mathbf{x}}. \tag{3.41}$$

Hence from the above and the equation (3.37)

$$\rho^n(t,\mathbf{x}) = \int_{\mathbb{R}^2_{\mathbf{v}}} f^n\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{v} \underset{n\to+\infty}{\longrightarrow} \rho(t,\mathbf{x}) = \int_{\mathbb{R}^2_{\mathbf{v}}} f\left(t,\mathbf{x},\mathbf{v}\right) d\mathbf{v} \qquad \mathbb{L}^1\left(]0,T_{max}[\times K_{\mathbf{x}}\right). \qquad \forall K_{\mathbf{x}}\, \text{compact}. \tag{3.42}$$

Step 4: Passing to the limit

According to the step (3),

$$\rho^n(t,\mathbf{x}) \underset{n\to+\infty}{\longrightarrow} \rho(t,\mathbf{x}) \qquad \mathbb{L}^1\left(]0,T_{max}[\times K_{\mathbf{x}}\right) \qquad \forall K_{\mathbf{x}}\, \text{compact}, \tag{3.43}$$

then, we can extract a sub-sequence again, as

$$\rho^n(t,\mathbf{x}) \underset{n\to+\infty}{\longrightarrow} \rho(t,\mathbf{x}) \qquad a.e\ [0,T_{max}[\times\mathbb{R}^2_{\mathbf{x}}. \tag{3.44}$$

On the other hand, by hypothesis we have:

$$||\mathbf{v}_{\mathbf{k}}^{n} - \mathbf{v}_d|| \underset{n\to+\infty}{\longrightarrow} ||\mathbf{v} - \mathbf{v}_d|| \qquad \forall \mathbf{k} \in \mathcal{K}. \tag{3.45}$$

Hence from (3.44), (3.45),

$$f_{eq}^{n} \underset{n\to+\infty}{\longrightarrow} f_{eq} \qquad a.e.\ [0, T_{max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}}. \tag{3.46}$$

Combine this results with those of the step (2) we have:

$$g = f_{eq}. \tag{3.47}$$

As a result, the left side of the equation (3.26) converges to $(f_{eq} - f)$ weakly in $\mathbb{L}^1\left(]0, T_{max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}}\right)$. We conclude that f is solution of the equation BGK (2.1).

4. Results and numerical simulations

4.1. **Numerical method.** The general idea of the semi-Lagrangian method used is to fix a grid in the velocity space and to transform the kinetic equation into a set of linear hyperbolic equations with source terms. We refer to [34, 35] for the detailed description of this numerical method. Here we recall only the basic principles. We summarize the semi-Lagrangian numerical method used in this work as follows:

(1) the discretization of the BGK model equation in the velocity space.
(2) A splitting procedure of time between transport and relaxation operators for each of the system evolution equations (4.1).
(3) The exact resolution of the transport part which means without using a spatial mesh, the initial data of this step are given by the solution of the relaxation operator.
(4) The resolution of the relaxation part on the grid with initial data defined by the value of the distribution function at the center of the cells after the transport step.

We introduce a Cartesian grid $\mathcal{V}$ in $\mathbb{R}^2$ in two-dimensional velocity space, and a set $\mathcal{K}$ multi−indices of $\mathbb{Z}^2$, such that:

$$\mathcal{K} = \{(-1,0); (-1,1); (0,1); (1,1); (1,0); (1,-1); (0,-1); (-1,-1)\}$$

which means we discretize the square $[-1,1] \times [-1,1]$.
In what follows, all the simulations are done in a square space domain $\Omega = [0,20] \times [0,20]$.
Thanks to the discrete velocity approximation above, the continuous distribution function f is replaced by 8 vector where each component is supposed to be an approximation of the distribution function f, i.e. $f_{\mathbf{k}}(t,\mathbf{x}) \approx f(t,\mathbf{x},\mathbf{v}_{\mathbf{k}})$, and the original kinetic equation (1.1) is replaced by a set of 8 evolution equations for $f_{\mathbf{k}}$ of the following form:

$$\begin{cases} \dfrac{\partial f_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \mathbf{v}_{\mathbf{k}}.\nabla_{\mathbf{x}} f_{\mathbf{k}}(t,\mathbf{x}) = \dfrac{1}{\tau}\left(f_{eq,\mathbf{k}}(\mathbf{v}_{\mathbf{k}}) - f_{\mathbf{k}}(t,\mathbf{x})\right) \qquad \mathbf{k} \in \mathcal{K}, \\ f_{\mathbf{k}}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) \qquad \mathbf{k} \in \mathcal{K}. \end{cases} \tag{4.1}$$

We describe the first step of the $[t^0 \to t^1]$ method, starting from $t^0 = 0$ and then we generalize at an arbitrary time step.
With splitting, the first step is reduced to the N_v linear transport equation resolution of the form

$$\frac{\partial f_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \mathbf{v}_{\mathbf{k}}.\nabla_{\mathbf{x}} f_{\mathbf{k}}(t,\mathbf{x}) = 0 \quad \mathbf{k} \in \mathcal{K}. \tag{4.2}$$

In order to solve this part, we consider for each equation of the system (4.2), the initial data defined by:

$$\widehat{f}_{\mathbf{k}}(\mathbf{x}, t=0) = f_{0,\mathbf{k}}(\mathbf{x}) \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \mathcal{K}. \tag{4.3}$$

Thanks to this reconstruction, the exact system solution (4.2) at time $t^1 = t^0 + \Delta t = \Delta t$ is given by:

$$\widehat{f}^*_{\mathbf{k}}(\mathbf{x}) = \widehat{f}_{\mathbf{k}}(\mathbf{x} - \mathbf{v_k}\Delta t) \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \mathcal{K}. \tag{4.4}$$

To complete a step of time, we must calculate the solution of the interaction part of the equation (4.1) on the points of the grid,

$$\partial_t f_{\mathbf{k}} = \frac{1}{\tau}\left(f_{eq,\mathbf{k}} - f_{\mathbf{k}}\right) \quad \mathbf{k} \in \mathcal{K}, \tag{4.5}$$

where the initial data is given by the resolution of the transport step to time $t^1 = t^0 + \Delta t$, $\left(\widehat{f}^*_{\mathbf{k}}(\mathbf{x})\right)_{\mathbf{k}\in\mathcal{K}}$.
To solve (4.5), we define the value of the equilibrium distribution at the instant t^1 ,$\left(f^1_{eq,\mathbf{k}}\right)_{\mathbf{k}\in\mathcal{K}}$,

$$f^1_{eq,\mathbf{k}}(\mathbf{v_k}) = \frac{\rho(t,\mathbf{x})}{2\pi v^2_{r,m,s}} exp\left(-\frac{1}{2}\frac{||\mathbf{v_k} - \mathbf{v}_d||^2}{v^2_{r,m,s}}\right), \quad \mathbf{k} \in \mathcal{K},$$

with

$$\rho(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} \widehat{f}^*_{\mathbf{k}}(\mathbf{x}).$$

Finally, the solution of relaxation equation (4.5) is given by:

$$f^1_{\mathbf{k}} = e^{\left(-\frac{\Delta t}{\tau}\right)} \widehat{f}^*_{\mathbf{k}} + \left(1 - e^{\left(-\frac{\Delta t}{\tau}\right)}\right) f^1_{eq,\mathbf{k}}, \ \mathbf{k} \in \mathcal{K}. \tag{4.6}$$

Given a density value $\left(\widehat{f}^n_{\mathbf{k}}(\mathbf{x})\right)_{\mathbf{k}\in\mathcal{K}}$ at the moment t^n for $\mathbf{x} \in \Omega$, $\mathbf{k} \in \mathcal{K}$, then the density at instant t^{n+1} can be calculated as follows:

- $\widehat{f}^*_{\mathbf{k}}(\mathbf{x}) = \widehat{f}^n_{\mathbf{k}}(\mathbf{x} - \mathbf{v_k}\Delta t)$.
- $f^{n+1}_{\mathbf{k}} = e^{\left(-\frac{\Delta t}{\tau}\right)} \widehat{f}^*_{\mathbf{k}} + \left(1 - e^{\left(-\frac{\Delta t}{\tau}\right)}\right) f^{n+1}_{eq,\mathbf{k}}$.

The local density at the instant t^{n+1} defined by:

$$\rho(t^{n+1},\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{n+1}_{\mathbf{k}}(\mathbf{x}).$$

By Referring to [34, 36, 37, **?**], this scheme is unconditionally stable , however for reasons of precision, the time step is chosen to satisfy the condition: $\frac{\Delta t}{\Delta x} < 1$ because the maximum speed of the pedestrians is fixed at one.
In conclusion, we summarize the procedures where the steps of this scheme in the following algorithm:

Require: $\left(f^0_{\mathbf{i}}(\mathbf{x})\right)_{\mathbf{i}\in\mathcal{K}}$: initial data.
for $m = 0 : Nt - 1$ **do**
• Resolution of the transport part the Nv equations of the system 4.2 with $(f^m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{K}}$ the initial data, we get $\left(\widehat{f}^m_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{K}}$.
• Computation of equilibrium density $\left(f^m_{eq,\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{K}}$.
• Relaxation term resolution (4.5) with $\left(\widehat{f}^m_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{K}}$ initial data.
end for
• $(f^m_{\mathbf{i}}(\mathbf{x}))_{\mathbf{i}\in\mathcal{K}}$.

The output parameters of the algorithm are the numerical solution of the original problem defined by the two equations (4.1) .

4.2. **Convergence test for the semi-Lagrangian scheme.** Our goal in this paper is to validate the proposed mathematical model for pedestrian motion. This choice is motivated by considering a small number of pedestrians ($N \leq 100$ pedestrians).
In order to analyze the convergence of the spatial discretization of the semi-Lagrangian method. We solve the kinetic equation (4.1), with initial data that is a piecewise constant function in two-dimensional space. They are concentrated densities in a center disk $\mathbf{x}_0$ with radius R_d:

$$\forall \mathbf{k} \in \mathcal{K}, \quad f_{\mathbf{k}}\left(t=0, \mathbf{x}\right) = \begin{cases} f^0_{\mathbf{k}}\left(\mathbf{x}\right), & ||\mathbf{x} - \mathbf{x}_0|| \leq R_d, \\ 0 & \end{cases} \tag{4.7}$$

This initial data corresponds to a small number of pedestrians $N = 100$.

We performed the simulation with a temporal discretization step, $\Delta t = 2.10^{-2}$.
In order to estimate the accuracy of the method, we used as a reference a solution f_* computed with a space step $\Delta x = \Delta y = \frac{1}{2^6} = 0.0156$, then we estimate the solution with different steps $\Delta x = \frac{1}{2^{n-1}}$, $n = 1, ..., 5$, with $\Delta x = \Delta y$. To evaluate the convergence, we calculated the difference between the norm $\mathbb{L}^1$ of the reference solution and the density $\rho(\mathbf{x})$ estimated at time $t = 2.5\,s$ with different steps Δx and Δy,

$$Error(\Delta x) = ||\rho(\Delta x, t = 2.5) - \rho_*(\Delta x, t = 2.5)||_{\mathbb{L}^1}, \tag{4.8}$$

with $\rho_*(., t = 2.5) = \sum_{\mathbf{k} \in \mathcal{K}} f_{\mathbf{k},*}(., t = 2.5)$.
We illustrate the results obtained in Fig.3.

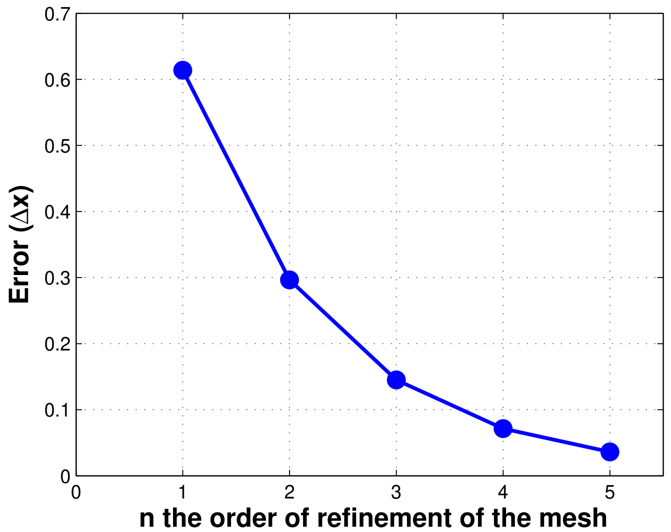


Figure 3. Precision of the schema in space with the error given by (4.8).

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The Fig. 3, shows that the norm of error $\mathbb{L}^1$ decreases linearly with respect Δx.
In order to demonstrate the convergence order, we compute the so called experimental order of convergence (EOC),

$$\text{EOC} := \log_2 \frac{||\rho(\Delta x, t = 2.5) - \rho_*||_{\mathbb{L}^1}}{||\rho(\frac{\Delta x}{2}, t = 2.5) - \rho_*||_{\mathbb{L}^1}}, \tag{4.9}$$

for considered spacial steps $\Delta x = \frac{1}{2^{n-1}}$, $n = 1, ..., 5$, the obtained results are given in the following table (see Table 1)

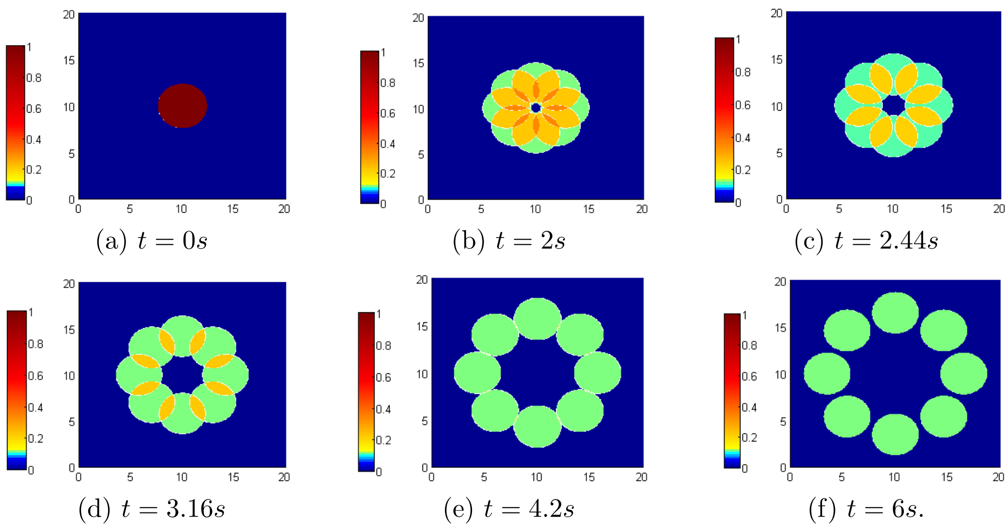
Table 1. Experimental order of convergence (EOC) for the semi-Lagrangian scheme.

n the order of refinement	$\Delta x = \frac{1}{2^{n-1}}$	Error(Δx)	EOC
1	1	0.6138	$--$
2	0.5000	0.2964	1.0502
3	0.2500	0.1450	1.0315
4	0.1250	0.0715	1.0200
5	0.0625	0.0362	0.9820

According to the results in Table 1, the experimental order of convergence EOC ≈ 1 showing that the method is first order accurate in space.
Then, to show the possible directions for pedestrians

$$\mathcal{K} = \Big\{(-1,-1), (-1,0); (-1,1); (1,-1); (1,0); (1,1); (0,1); (0,-1)\Big\},$$

$\mathcal{V} = \{\mathbf{v_k} = \mathbf{k}\Delta v \ \ /\mathbf{k} \in \mathcal{K},\ \Delta v = 1\}$, we represented the solution at 6 different moments, with the same initial condition defined by (4.7), and space step $\Delta x = \Delta y = 5.10^{-2}$ and $\Delta t = 2.10^{-2}$, the results obtained are represented in Fig.4 a-f.



(a) $t = 0s$ (b) $t = 2s$ (c) $t = 2.44s$
(d) $t = 3.16s$ (e) $t = 4.2s$ (f) $t = 6s$.

Figure 4. The evolution of the density $\rho(x, y)$ during the instants, (a): $t = 0s$, (b): $t = 2s$, (c): $t = 2.44s$ (d): $t = 3.16s$, (e): $t = 4.2s$, (f): $t = 6s$.

We observe that pedestrian density diffuses into space and pedestrians point to the following directions:

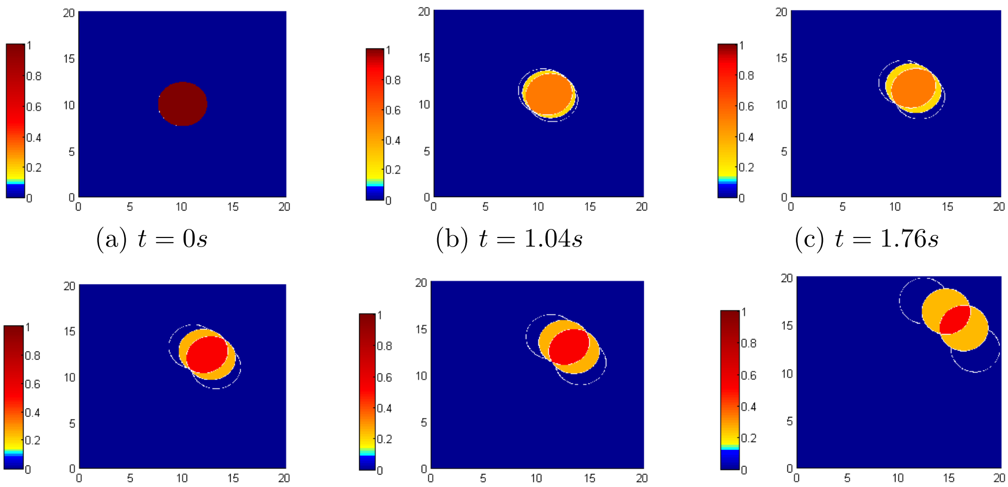
$$\Big\{(\pm 1, 0); (1, \pm 1); (-1, \pm 1); (0, \pm 1)\Big\}.$$

4.3. **Movement of a group of pedestrians towards a single desired direction $\mathbf{v}_d$.** In this paragraph our aim is to show the adaptation of all pedestrians to a desired configuration namely the movement in a desired direction $\mathbf{v}_d$. We consider the BGK model (4.1) with the same initial condition (4.7) corresponding to 100 pedestrians. The equilibrium density modeling the trend towards the desired configuration is defined as follows:

$$\text{(4.10)} \qquad \begin{cases} f_{eq,\mathbf{k}}(\mathbf{v_k}) = \dfrac{\rho(t,\mathbf{x})}{2\pi v_{r,m,s}^2} exp\left(-\dfrac{1}{2}\dfrac{||\mathbf{v}_k - \mathbf{v}_d||^2}{v_{r,m,s}^2}\right), \quad \mathbf{k} \in \mathcal{K} \ \mathbf{v}_d = [1,1]^T, \\ \mathcal{K} = \{(-1,0);(1,-1);(1,0);(1,1);(0,1)\} \ \mathbf{v_k} \in \mathcal{V} = \{\mathbf{v_k} = \mathbf{k}\Delta v \ \ /\mathbf{k} \in \mathcal{K}, \, \Delta v = 1\}. \end{cases}$$

We assume that all pedestrians have the same speed $v = 1.00$ m $/s$. We considered the same time step as in the first case $\Delta t = 2.10^{-2}$ $\Delta x = \Delta y = 5.10^{-2}$.

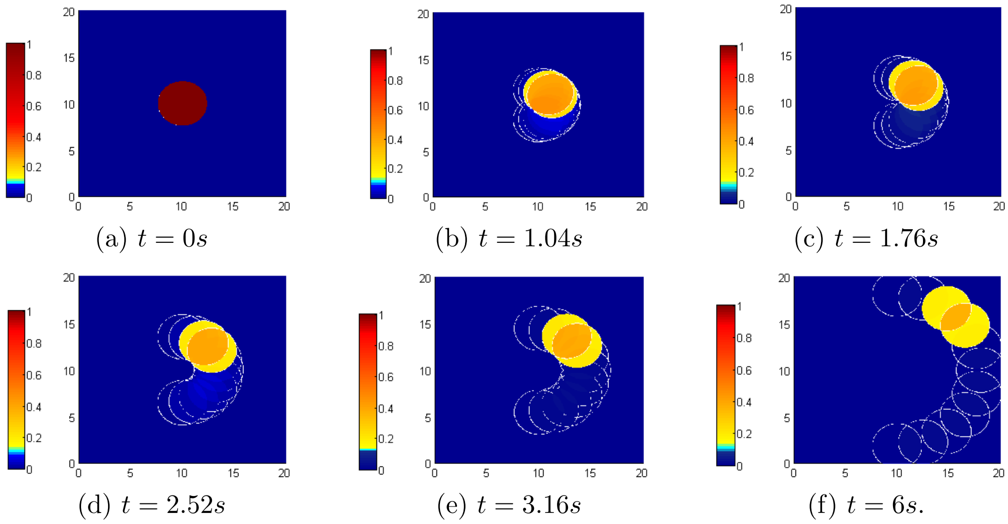
We then represent the evolution of pedestrian density for 6 different instants and for two values of relaxation time $\tau = 5.10^{-3}$ (see Fig.5) and $\tau = 5.10^{-2}$ (see Fig.6.)



(a) $t = 0s$ (b) $t = 1.04s$ (c) $t = 1.76s$

(d) $t = 2.52s$ (e) $t = 3.16s$ (f) $t = 6s$.

Figure 5. Evolution of the local density during the times : (a): $t = 0s$, (b): $t = 1.04s$, (c): $t = 1.76s$ (d): $t = 2.52.16s$, (e): $t = 3.16s$, (f): $t = 6s$, with a relaxation time $\tau = 5.10^{-3}$.



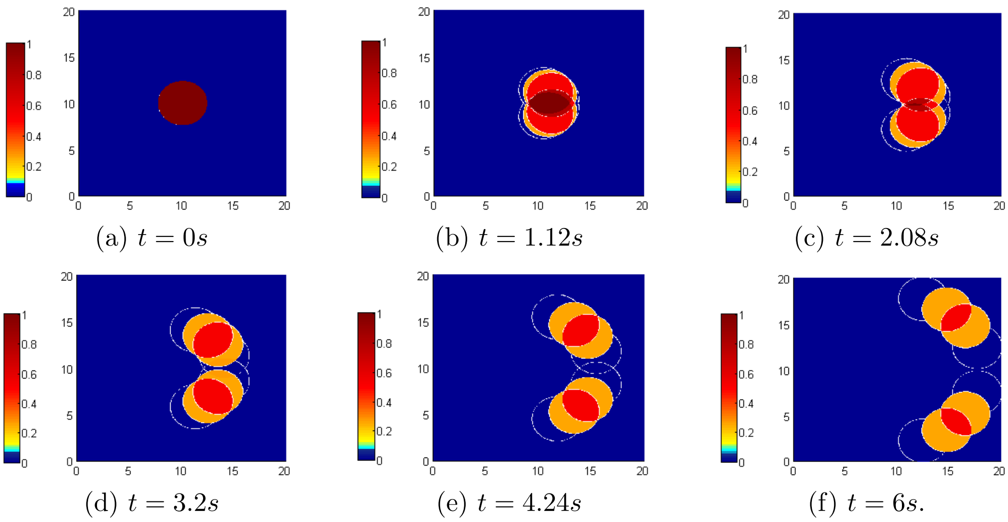
(a) $t = 0s$ (b) $t = 1.04s$ (c) $t = 1.76s$

(d) $t = 2.52s$ (e) $t = 3.16s$ (f) $t = 6s$.

Figure 6. Evolution of the local density during the times : (a): $t = 0s$, (b): $t = 1.04s$, (c): $t = 1.76s$ (d): $t = 2.52.16s$, (e): $t = 3.16s$, (f): $t = 6s$, with a relaxation time $\tau = 5.10^{-2}$.

At the moment $t = 0$, we have the possible directions $\{(\pm 1, 0); (1, \pm 1); (-1, \pm 1); (0, \pm 1)\}$. Then all pedestrians have the tendency to direct towards a desired direction $\mathbf{v}_d = [1, 1]^T$. Firstly, the figures show the adaptation of the desired direction $\mathbf{v}_d = [1, 1]^T$ by the pedestrian group, on the other hand the figure 5 shows that the time taken by the group of pedestrians that they have a relaxation time $\tau = 5.10^{-3}$ to direct towards $\mathbf{v}_d$ is minimal compared to the time taken by the group who have $\tau = 5.10^{-2}$ fig. 6.

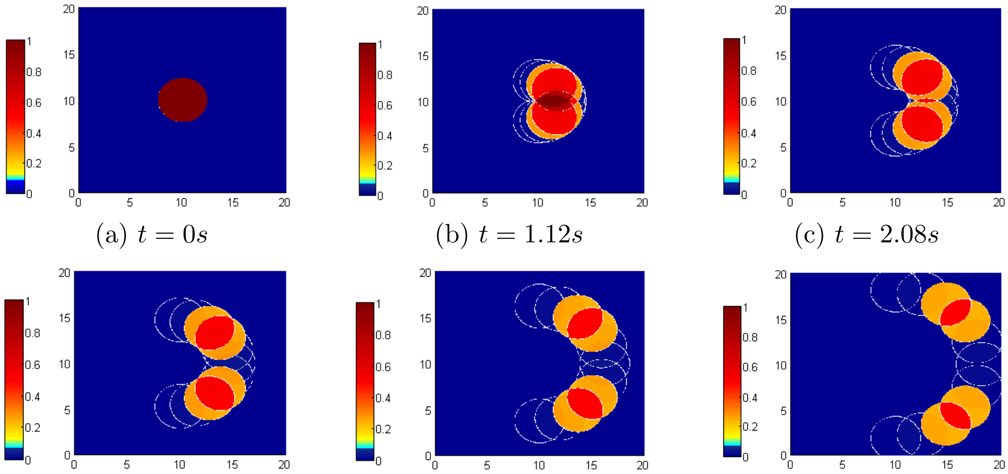
4.4. **Motion of a group of pedestrians towards 2 desired directions $\mathbf{v}_{d,1}, \mathbf{v}_{d,2}$.** Consider a group of pedestrians defined by the same initial data. This initial density corresponds to a number of pedestrians equal to 100 pedestrians have the tendency to direct towards two desired directions $\mathbf{v}_{d,1} = [1, 1], \mathbf{v}_{d,2} = [-1, 1]$, all pedestrians have the same speed $v = 1.00\, m/s$.
The results obtained are given in the figure 7 for $\tau = 5.10^{-3}$ and the figure 8 for $\tau = 5.10^{-2}$.



(a) $t = 0s$ (b) $t = 1.12s$ (c) $t = 2.08s$

(d) $t = 3.2s$ (e) $t = 4.24s$ (f) $t = 6s$.

Figure 7. Evolution of local density during the times : (a): $t = 0s$, (b): $t = 1.12s$, (c): $t = 2.08s$ (d): $t = 3.2s$, (e): $t = 4.24s$, (f): $t = 6s$, with a relaxation time $\tau = 5.10^{-3}$.



(a) $t = 0s$ (b) $t = 1.12s$ (c) $t = 2.08s$

(d) $t = 3.2s$ (e) $t = 4.24s$ (f) $t = 6s$.

Figure 8. Evolution of local density during the times : (a): $t = 0s$, (b): $t = 1.12s$, (c): $t = 2.08s$ (d): $t = 3.2s$, (e): $t = 4.24s$, (f): $t = 6s$, with a relaxation time $\tau = 5.10^{-2}$.

The figures show that the crowd is divided into two groups in order to reach their state of equilibrium. It consists the movement of the two groups towards the two desired directions $\mathbf{v}_{d,1}, \mathbf{v}_{d,2}$.

5. Conclusion and looking forward

In this paper we have developed the kinetic approach for the dynamics of a crowd, based on the BGK equation. The existence and uniqueness of the proposed discrete velocity model solution has been demonstrated thanks to the Banach fixed point theorem. Thus, the convergence of this discrete model towards the continuous BGK model is proven. Numerical simulations using the semi-Lagrangian method are performed. The mathematical model proposed is capable of describing the tendency of a crowd towards a situation of equilibrium, namely the tendency towards a desired direction.

As already laid out in the introduction, it should be clear that the aim of this paper is only to adapt the BGK model in kinetic theory to the movement of a crowd. In other words, the derivation of the equilibrium function f_{eq} describing a pedestrian tendency to achieve a desired direction. The most important perspective, is related to the derivation of a equilibrium function describing the panic conditions, evacuation problems or lane formation. The relaxation time τ is assumed constant this is a strong assumption considering the application. This coefficient τ may depend on f_{eq} or on the density ρ.

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