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ANALYSIS OF THE NONLOCAL WAVE PROPAGATION PROBLEM WITH VOLUME CONSTRAINTS

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Abstract. In the current paper, we investigate a nonlocal hyperbolic problem with volume constraints. The main motivation of this work is to apply the nonlocal vector calculus, introduced and developed by DU et al. [3] to such problem. Moreover, based on some density arguments, some a priori estimates and using the Galerkin approach, we prove existence and uniqueness of a weak solution to the nonlocal wave equation.

1. **Introduction.** The study of nonlocal problems has gained great attention over the last two decades. Nonlocal models involve integral equations and fractional derivatives allowing nonlocal interactions, that is to say, the interaction may occur even when the closures of two domains have an empty intersection. Such models are effective in modeling material singularities and are widely considered in a variety of applications, including image analyses [6]-[10], phase transition [4][11], machine learning [12] and obstacle problem [5]...
In a major advance in 2013, Du et al. [3] introduced nonlocal vector calculus as a nonlocal framework to understand and analyze nonlocal problems. It defines nonlocal fluxes , nonlocal analogues of the gradient, divergence, and curl operators, and presentes some basic nonlocal integral theorems that mimic the classical integral theorems of the vector calculus for differential operators, the authors have also provided connection between the nonlocal operators and their usual differential counterparts in a distributional sense then in a weak sense by introducing nonlocal weighted operators.
The present paper was motivated by [2], where the authors threw light on the analogy between nonlocal and local diffusion problems with a convincing explanation of the usefulness, in the nonlocal case, of volume constraints which represent the nonlocal analogue of the boundary conditions of the classical theory. Our purpose

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is to discuss the well posedeness of a hyperbolic problem considering a nonlocal diffusion operator instead of the Laplacian operator. Furthermore, the study of the eigenvalue problem corresponding to the nonlocal Dirichlet problem is carried out. The paper is divided into two main sections. The first part gives a brief overview of the basic concepts of the nonlocal vector calculus and emphasises the existence of an orthogonal basis of eigenfunctions associated to the considered nonlocal operator. In the second part, we formulate the nonlocal wave equation and exploit the Galerkin method to prove existence and uniqueness of weak solution to the nonlocal hyperbolic problem.

2. **Statement of the elliptic nonlocal problem.** In the present section we give the position of the elliptic volume constrained problem and present the energy spaces needed to study the nonlocal problem:

$$\begin{cases} \mathcal{D}(\xi.\mathcal{D}^*(u)) & = & f & \text{on} \quad \Omega \\ u & = & 0 & \text{on} \quad \Omega_I \end{cases} \tag{1}$$

Where Ω is an open and bounded subset of $\mathbb{R}^n$ with piecewise smooth boundary and satisfies the interior cone condition, ξ is a symmetric second-order tensor, and $f \in L^2(\Omega)$ is a given function.
Recall the definition of some nonlocal operators, see [3]. Given a vector function $\nu(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ and an antisymmetric vector function $\alpha(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, the action of the nonlocal divergence operator $\mathcal{D}$ on ν is defined as

$$\mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x,y) + \nu(y,x)).\alpha(x,y)dy \quad for\ x \in \mathbb{R}^n \tag{2}$$

Given a scalar function $u(x) : \mathbb{R}^n \to \mathbb{R}$, the adjoint of $\mathcal{D}$ is the operator $\mathcal{D}^*$ whose action on u is given by

$$\mathcal{D}^*(u)(x,y) = -(u(y) - u(x))\alpha(x,y) \quad for\ x, y \in \mathbb{R}^n \tag{3}$$

The operator $-\mathcal{D}^*$ is considered as a nonlocal gradient, also,

$$\mathcal{D}(\xi.\mathcal{D}^* u)(x) = -2\int_{\mathbb{R}^n} (u(y) - u(x))\alpha(x,y).(\xi(x,y).\alpha(x,y))dy$$

Given positive constants γ_0 and ε, we first assume that the symmetric kernel

$$\gamma(x,y) = \alpha(x,y).(\xi(x,y).\alpha(x,y))$$

satisfies, for all $x \in \Omega \cup \Omega_I$
1) $\gamma(x,y) \geq 0 \quad \forall y \in B_\varepsilon(x)$
2) $\gamma(x,y) \geq \gamma_0 > 0 \quad \forall y \in B_{\varepsilon/2}(x)$
3) $\gamma(x,y) = 0 \quad \forall y \in (\Omega \cup \Omega_I) \setminus B_\varepsilon(x)$
where $B_\varepsilon(x) := \{y \in \Omega \cup \Omega_I \ : \ |y - x| \leq \varepsilon\}$
4) There exist $s \in (0,1)$ and positive constants γ_* and γ^* such that, for all $x \in \Omega$

$$\frac{\gamma_*}{|y-x|^{n+2s}} \leq \gamma(x,y) \leq \frac{\gamma^*}{|y-x|^{n+2s}} \quad for\ y \in B_\varepsilon(x)$$

let us also recall the definition of the interaction domain corresponding to Ω:

$$\Omega_I := \{y \in \mathbb{R}^n \backslash \Omega \ : \ \alpha(x,y) \neq 0 \text{ for some } x \in \Omega\} \tag{4}$$

To investigate the problem (1), the following nonlocal energy space will be used constantly [2]. We adopt:

$$V(\Omega \cup \Omega_I) := \{u \in L^2(\Omega \cup \Omega_I) \ : \ |||u||| < \infty\} \tag{5}$$

equipped with the nonlocal energy norm

$$|||u||| := \left(\frac{1}{2}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(u))dydx\right)^{\frac{1}{2}} \tag{6}$$

we then introduce the nonlocal volume constrained energy space [2]:

$$V_c(\Omega\cup\Omega_I) := \{u \in V(\Omega\cup\Omega_I) \ : \ u = 0 \ \ on \ \Omega_I\}$$

the norm

$$|||f|||_{V_c^*(\Omega\cup\Omega_I)} := \sup_{\substack{\varphi\in V_c(\Omega\cup\Omega_I)\\ |||\varphi|||\leq 1}} \left|\langle f,\varphi\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}\right|$$

denotes the norm for the dual space $V_c^*(\Omega\cup\Omega_I)$ of $V_c(\Omega\cup\Omega_I)$.
Next, using the nonlocal Green's first identities [3], we state the following definition:

Definition 2.1. We say that $u \in V_c(\Omega\cup\Omega_I)$ is a weak solution of the nonlocal elliptic problem (1) if

$$\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx = \int_\Omega f\varphi dx \qquad \forall\varphi\in V_c(\Omega\cup\Omega_I). \tag{7}$$

Then, according to the definition of the nonlocal energy norm (6), we immediately announce the following theorem:

Theorem 2.2. *There exist two constants $M_1, M_2 > 0$ such that*

$$\left|\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx\right| \leq M_1|||u||||||\varphi||| \tag{8}$$

and

$$M_2|||u|||^2 \leq \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(u))dydx \tag{9}$$

for all $u,\varphi\in V_c(\Omega\cup\Omega_I)$

Theorem 2.3. *For each $f\in L^2(\Omega)$, there exists a unique weak solution $u\in V_c(\Omega\cup\Omega_I)$ of the nonlocal elliptic problem (1).*

Proof. Using the previous theorem (2.2), we obtain the result of existence and uniqueness via a direct application of the Lax-Milgram theorem. □

2.1. **The nonlocal Dirichlet eigenvalue problem.** In this subsection, we focus our attention on seeking the set of numbers μ such that the following eigenvalues problem (10) corresponding to the Dirichlet nonlocal problem (1):

$$\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx = \mu\int_\Omega u\varphi dx \qquad \forall\varphi\in V_c(\Omega\cup\Omega_I), \tag{10}$$

has a solution $u\in V_c(\Omega\cup\Omega_I)$.
We state the following result:

Theorem 2.4. 1) *Each eigenvalue of the problem (10) is real.*
2) *If we repeat each eigenvalue according to its multiplicity, we have that the set Σ of the eigenvalues of the operator $\mathcal{D}(\xi.\mathcal{D}^*(.))$ is as follows:*

$$\Sigma = (\mu_j)_{j\geq 1} \tag{11}$$

where $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq ...$ and $\mu_j \underset{j\to\infty}{\to} \infty$.
3) *There exists an orthonormal basis $(v_j)_{j\geq 1}$ of $L^2(\Omega\cup\Omega_I)$, where $v_j\in V_c(\Omega\cup\Omega_I)$ is an eigenvector corresponding to μ_j for $j\geq 1$.*

Proof. Let K be the mapping

$$\begin{array}{rcl} K : L^2(\Omega \cup \Omega_I) & \to & V_c(\Omega \cup \Omega_I) \\ f & \mapsto & u_f \end{array}$$

where u is the unique solution of (1) given by Theorem (2.3).
We claim that the operator K is bounded, indeed:

$$\begin{array}{rcl} |||u|||^2 & = & \displaystyle\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u).(\xi.\mathcal{D}^*(u))dydx \\ & \leq & \| f \|_{L^2(\Omega)} \| u \|_{L^2(\Omega)} \end{array}$$

according to the nonlocal Poincaré inequality [2], there exists a positive constant C such that:

$$|||Kf||| \leq \| f \|_{L^2(\Omega\cup\Omega_I)}$$

since the embedding

$$\begin{array}{rcl} I : V_c(\Omega \cup \Omega_I) & \to & L^2(\Omega \cup \Omega_I) \\ u & \mapsto & u \end{array}$$

is compact, we directly deduce that the mapping

$$\begin{array}{rcl} I \circ K : L^2(\Omega \cup \Omega_I) & \to & L^2(\Omega \cup \Omega_I) \\ f & \mapsto & u_f \end{array}$$

is linear and compact.
On the other hand, if w is the unique solution of the problem:

$$\left\{ \begin{array}{lcll} D(\xi.D^*(w)) & = & f & \text{on} \quad \Omega \\ w & = & 0 & \text{on} \quad \Omega_I \end{array} \right.$$

and v if the solution of:

$$\left\{ \begin{array}{lcll} D(\xi.D^*(v)) & = & g & \text{on} \quad \Omega \\ v & = & 0 & \text{on} \quad \Omega_I \end{array} \right.$$

where $f, g \in L^2(\Omega \cup \Omega_I)$. We have:

$$((I \circ K)f, g)_{L^2(\Omega\cup\Omega_I)} = \int_\Omega wgdx = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(v).(\xi.\mathcal{D}^*(w))dydx$$

$$((I \circ K)g, f)_{L^2(\Omega\cup\Omega_I)} = \int_\Omega vfdx = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(w).(\xi.\mathcal{D}^*(v))dydx$$

which prove that the operator $I \circ K$ is symmetric. In addition:

$$((I \circ K)f, f)_{L^2(\Omega\cup\Omega_I)} = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(w).(\xi.\mathcal{D}^*(w))dydx \geq 0$$

We apply the theory of compact and symmetric operators from [13] to conclude the existence of real eigenvalues of $I \circ K$, and that the corresponding eigenvectors $(v_j)_{j\geq 1}$ form a complete orthonormal system in $L^2(\Omega \cup \Omega_I)$.
To conclude the proof, notice that:

$$(I \circ K)v = \lambda v \text{ is equivalent to } \mathcal{D}(\xi.\mathcal{D}^*(v)) = \frac{1}{\lambda}v = \mu v$$

□

Theorem 2.5. *Let* $(v_j)_{j\geq 1}$ *be the eigenvectors corresponding to* $(\mu_j)_{j\geq 1}$ *given by Theorem (2.4), then* $(v_j)_{j\geq 1}$ *forms an orthogonal basis of* $V_c(\Omega\cup\Omega_I)$.

Proof. The orthogonality of the eigenvectors follows from:

$$\begin{aligned}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(v_j).(\xi.\mathcal{D}^*(v_k))dydx &= \mu_j(v_j,v_k)_{L^2(\Omega)}\\ &= \mu_j\delta_{i,j}\end{aligned}$$

On the other hand, for each $u\in V_c(\Omega\cup\Omega_I)$ we have:

$$\begin{aligned}u &= \sum_{j\geq 1}(u,v_j)_{L^2(\Omega\cup\Omega_I)}v_j\\ &= \sum_{j\geq 1}\frac{\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(v_j))dydx}{\mu_j}v_j\\ &= \sum_{j\geq 1}\frac{\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(v_j))dydx}{|||v_j|||^2}v_j\end{aligned}$$

which concludes the proof. □

Since $(v_j)_{j\geq 1}$ is an orthogonal basis of $V_c(\Omega\cup\Omega_I)$ for any $j\in\mathbb{N}$, we can define the orthogonal projection on the j-dimensional subspace of $V_c(\Omega\cup\Omega_I)$ spanned by $v_1, v_2, ..., v_j$.

Proposition 1. *Let* P_n*,* Q_n *be the orthogonal projections defined, for all* $n\in\mathbb{N}$*, by:*

$$P_n(u) := \sum_{j=1}^{n}(u,v_j)_{L^2(\Omega\cup\Omega_I)}v_j \qquad \forall u\in L^2(\Omega\cup\Omega_I) \tag{12}$$

$$Q_n(u) := \sum_{j=1}^{n}\frac{\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(v_j))dydx}{|||v_j|||^2}v_j \qquad \forall u\in V_c(\Omega\cup\Omega_I) \tag{13}$$

Then

$$P_n u \xrightarrow{L^2(\Omega\cup\Omega_I)} u \qquad \forall u\in L^2(\Omega\cup\Omega_I)$$

Then

$$Q_n u \xrightarrow{V_c(\Omega\cup\Omega_I)} u \qquad \forall u\in V_c(\Omega\cup\Omega_I)$$

These convergences come simply from the following result:

Proposition 2. *Let* P_n, Q_n *be the orthogonal projections defined by definitions (12) and (13), then:*

$$\| P_n \|_{\mathcal{L}(L^2(\Omega\cup\Omega_I),L^2(\Omega\cup\Omega_I))} = \| Q_n \|_{\mathcal{L}(V_c(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I))} = 1$$

If we set

$$P_n(u) = \sum_{j=1}^{n} < u, v_j >_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)} v_j \qquad \forall u\in V_c^*(\Omega\cup\Omega_I)$$

then

$$\| P_n \|_{\mathcal{L}(V_c^*(\Omega\cup\Omega_I),V_c^*(\Omega\cup\Omega_I))} = 1$$

Proof. Let $u \in L^2(\Omega \cup \Omega_I)$, then

$$\| u \|^2_{L^2(\Omega\cup\Omega_I)} = \lim_{n\to\infty} \sum_{j=1}^{n} (u, v_j)^2_{L^2(\Omega\cup\Omega_I)} = \lim_{n\to\infty} \| P_n(u) \|^2_{L^2(\Omega\cup\Omega_I)}$$

subsequently

$$\| P_n \|_{\mathcal{L}(L^2(\Omega\cup\Omega_I), L^2(\Omega\cup\Omega_I))} = \sup_{\substack{u\in L^2(\Omega\cup\Omega_I)\\ u\neq 0}} \frac{\| P_n(u) \|_{L^2(\Omega\cup\Omega_I)}}{\| u \|_{L^2(\Omega\cup\Omega_I)}} \leq 1$$

to conclude the proof, note that: $P_n(v_j) = v_j$
Secondly, we have, for $u \in V_c(\Omega \cup \Omega_I)$:

$$\begin{aligned} Q_n(u) &= \sum_{j=1}^{n} \frac{\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u).(\xi.\mathcal{D}^*(v_j))dydx}{|||v_j|||^2} v_j \\ &= \sum_{j=1}^{n} (u, v_j)_{L^2(\Omega\cup\Omega_I)} v_j \end{aligned}$$

therefore

$$\begin{aligned} |||Q_n(u)||| &= \sum_{j=1}^{n} (u, v_j)^2_{L^2(\Omega\cup\Omega_I)} \frac{\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(v_j).(\xi.\mathcal{D}^*(v_j))dydx}{|||v_j|||^2} v_j \\ &\leq \sum_{j=1}^{\infty} (u, v_j)^2_{L^2(\Omega\cup\Omega_I)} |||v_j|||^2 \\ &\leq |||u|||^2 \end{aligned}$$

and as $Q_n(v_j) = v_j$, we claim that $\| Q_n \|_{\mathcal{L}(V_c(\Omega\cup\Omega_I), V_c(\Omega\cup\Omega_I))} = 1$.
Furthermore, if we extend the projection P_n to $V_c^*(\Omega \cup \Omega_I)$ we obtain:

$$\begin{aligned} | < P_n(u), \varphi >_{V_c^*(\Omega\cup\Omega_I), V_c(\Omega\cup\Omega_I)} | &= | \sum_{j=1}^{n} < u, v_j >_{V_c^*(\Omega\cup\Omega_I), V_c(\Omega\cup\Omega_I)} (\varphi, v_j)_{L^2(\Omega\cup\Omega_I)} | \\ &= | < u, P_n(\varphi) >_{V_c^*(\Omega\cup\Omega_I), V_c(\Omega\cup\Omega_I)} | \\ &\leq \| u \|_{V_c^*(\Omega\cup\Omega_I)} |||\varphi||| \end{aligned}$$

hence

$$\| P_n(u) \|_{V_c^*(\Omega\cup\Omega_I)} \leq \| u \|_{V_c^*(\Omega\cup\Omega_I)}$$

□

3. **The nonlocal wave equation.**

3.1. **statement of the problem.** We denote by Ω an open set of $\mathbb{R}^n$ and Ω_I its corresponding interaction domain. We will always assume that Ω and Ω_I are bounded with piecewise smooth boundary and satisfy the interior cone condition. The example of nonlocal hyperbolic equation that we consider is the following: we seek a real valued function $u = u(x,t)$, $x \in \Omega$, $t \in]0,T]$, solution to

$$\begin{cases} u_{tt} + \mathcal{D}(\xi.\mathcal{D}^*(u)) = f & \text{in } \Omega\times]0,T] \\ u = 0 & \text{on } \Omega_I\times]0,T] \\ u(x,0) = g(x) & \text{on } \Omega \\ u_t(x,0) = h(x) & \text{on } \Omega \end{cases} \tag{14}$$

Where $\mathcal{D}$, $\mathcal{D}^*$ are, respectively, the nonlocal divergence (2) and the nonlocal gradient (3), $\xi(x,y)$ denotes a symmetric, positive definite second order tensor having elements that are symmetric functions of x and y and $f:\Omega\times]0,T[\to\mathbb{R},\ g,\ h:\Omega\to\mathbb{R}$ are given.
First, we specify in which sense we want to solve the problem. (14)

Definition 3.1. If $f\in L^2(0,T;L^2(\Omega))$, $g\in V_c(\Omega\cup\Omega_I)$ and $h\in L^2(\Omega\cup\Omega_I)$ we say a function $u\in L^2(0,T;V_c(\Omega\cup\Omega_I))$ with $u^{'}\in L^2(0,T;L^2(\Omega\cup\Omega_I))$ and $u^{''}\in L^2(0,T;V_c^*(\Omega\cup\Omega_I))$ is a weak solution of the nonlocal constrained problem (14) if

$$\langle u^{''},\varphi\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx=\int_\Omega f\varphi dx$$

$\forall\varphi\in V_c(\Omega\cup\Omega_I)$ and a.e $0\le t\le T$, with $u(0)=g$, $u^{'}(0)=h$

3.2. **Galerkin approximation.** Let $(v_j)_{j\ge1}$ be the eigenvectors corresponding to the eigenvalues $(\lambda_j)_{j\ge1}$ of the problem (10), given by Theorem (2.4).
For a fixed $n\ge1$, we are looking for a function $u_n:[0,T]\to V_c(\Omega\cup\Omega_I)$ of the form

$$u_n(t):=\sum_{j=1}^n p_{jn}(t)v_j \tag{15}$$

such that

$$\begin{cases} j=1,...,n \\ \langle u_n^{''},v_j\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(v_j))dydx=\int_\Omega f_nv_jdx \\ u_n(0)=\sum_{j=1}^n(g,v_j)_{L^2(\Omega)}v_j \\ u_n^{'}(0)=\sum_{j=1}^n(h,v_j)_{L^2(\Omega)}v_j \end{cases} \tag{16}$$

where $(f_n)_n\in\mathcal{D}(\Omega\times(0,T))$ such that $f_n\xrightarrow{L^2(\Omega\times(0,T))}f$
with $\|f_n\|_{L^2(\Omega\times(0,T))}\le\|f\|_{L^2(\Omega\times(0,T))}$

Theorem 3.2. *For each integer $n\ge1$ there exists a unique function u_n of the form (15) satisfying (16).*

Proof. To solve the problem (16), we shall find

$$p_n(t)=(p_{1n}(t),p_{2n}(t),...,p_{nn}(t))\in\mathbb{R}^n$$

solution to

$$\begin{cases} j=1,...,n \\ p_{jn}^{''}+\sum_{k=1}^n\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(v_j).(\xi.\mathcal{D}^*(v_k))p_{kn}(t)dydx=\int_\Omega f_nv_jdx \\ p_{jn}(0)=(g,v_j)_{L^2(\Omega)} \\ p_{jn}^{'}(0)=(h,v_j)_{L^2(\Omega)} \end{cases} \tag{17}$$

According to standard existence theory for ODE, there exists a unique function

$$p_n(t)=(p_{1,n}(t),p_{2,n}(t),...,p_{n,n}(t)) \tag{18}$$

satisfying (17) for a.e $0\le t\le T$. □

3.3. **Energy estimates.** In order to show that $(u_n)_{n\geq 1}$ converges to a weak solution of (14) we will need some uniform estimates.

Theorem 3.3. *There exists a positive constant M such that*

$$\max_{0\leq t\leq T}\left(|||u_n(t)|||+\parallel u_n^{'}(t)\parallel_{L^2(\Omega\cup\Omega_I)}\right)+\parallel u_n^{''}\parallel_{L^2(0,T;V_c^*(\Omega\cup\Omega_I))}$$

$$\leq M\left(\parallel f\parallel_{L^2(0,T;L^2(\Omega))}+|||g|||+\parallel h\parallel_{L^2(\Omega)}\right)\quad for\ n\geq 1$$

Proof. We multiply equation (16) by $p_{jn}^{'}(t)$, sum for $j=1,...,n$, we find

$$\langle u_n^{''},u_n^{'}\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(u_n^{'}))dydx=\int_\Omega f_n u_n^{'}dx$$

for a.e $0\leq t\leq T$.
which give

$$\begin{aligned}\frac{d}{dt}\left(\parallel u_n^{'}\parallel^2_{L^2(\Omega\cup\Omega_I)}+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(u_n))dydx\right) &\leq 2\parallel f_n\parallel_{L^2(\Omega)}\parallel u_n^{'}\parallel_{L^2(\Omega)}\\ &\leq \parallel f_n\parallel^2_{L^2(\Omega)}+\parallel u_n^{'}\parallel^2_{L^2(\Omega\cup\Omega_I)}+|||u_n|||^2\end{aligned}$$

Gronwall's inequality and the proposition (2) imply

$$\begin{aligned}\parallel u_n^{'}\parallel^2_{L^2(\Omega\cup\Omega_I)}+|||u_n|||^2 &\leq M\left(|||P_n(g)|||^2+\parallel P_n(h)\parallel^2_{L^2(\Omega\cup\Omega_I)}+\parallel f_n\parallel_{L^2(0,T;L^2(\Omega))}\right)\\ &\leq M\left(|||g|||^2+\parallel h\parallel^2_{L^2(\Omega\cup\Omega_I)}+\parallel f_n\parallel^2_{L^2(0,T;L^2(\Omega))}\right)\end{aligned}$$

as $0\leq t\leq T$ was chosen arbitrarily, we obtain:

$$\max_{0\leq t\leq T}\left(\parallel u_n^{'}(t)\parallel^2_{L^2(\Omega\cup\Omega_I)}+|||u_n|||^2\right)\leq M\left(|||g|||^2+\parallel h\parallel^2_{L^2(\Omega\cup\Omega_I)}+\parallel f\parallel^2_{L^2(0,T;L^2(\Omega))}\right)\tag{19}$$

To conclude, we fix any $\varphi\in V_c(\Omega\cup\Omega_I)$ with

$$|||\varphi|||\leq 1\quad\text{and}\quad\varphi=P_n(\varphi)+\psi$$

where $(\psi,v_j)_{L^2(\Omega\cup\Omega_I)}=0\qquad for\ 1\leq j\leq n$, we get:

$$\langle u_n^{''},\varphi\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}=\int_\Omega f_nP_n(\varphi)dx-\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(P_n(\varphi)))dydx$$

Consequently, since $|||P_n(\varphi)|||\leq 1$

$$|\langle u_n^{''},\varphi\rangle|_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}\leq M(\parallel f_n\parallel^2_{L^2(\Omega)}+|||u_n|||)$$

finally, using (19) we get

$$\left\|u_n^{''}\right\|_{L^2(0,T;V_c^*(\Omega\cup\Omega_I))}\leq M(\parallel f\parallel^2_{L^2(0,T;L^2(\Omega))}+|||g|||^2+||h||^2_{L^2(\Omega)})$$

□

3.4. **Existence and uniqueness result.**

Theorem 3.4. *The nonlocal parabolic problem* (14) *admits a unique weak solution.*

Proof. Using the previous Theorem (3.3), we conclude that $(u_n)_{n\geq 1}$ is bounded in $L^2(0,T;V_c(\Omega\cup\Omega_I))$, with $(u_n^{'})_{n\geq 1}$ is bounded in $L^2(0,T;L^2(\Omega\cup\Omega_I))$, and $(u_n^{''})_{n\geq 1}$ is bounded in $L^2(0,T;V_c^*(\Omega\cup\Omega_I))$.
Consequently, there exists a subsequence still denoted $(u_n)_{n\geq 1}$ and a function $u\in L^2(0,T;V_c(\Omega\cup\Omega_I))$ with $u^{'}\in L^2(0,T;L^2(\Omega\cup\Omega_I))$ and $u^{''}\in L^2(0,T;V_c^*(\Omega\cup\Omega_I))$, such that:

$$\begin{cases} u_n & \rightharpoonup & u \quad \text{in } L^2(0,T;V_c(\Omega\cup\Omega_I) \\ u_n^{'} & \rightharpoonup & u^{'} \quad \text{in } L^2(0,T;L^2(\Omega\cup\Omega_I) \\ u_n^{''} & \rightharpoonup & u^{''} \quad \text{in } L^2(0,T;V_c^*(\Omega\cup\Omega_I) \end{cases} \tag{20}$$

next, fix an integer N and select $n\geq N$, choose a function $\psi\in L^2(0,T)$ and $\varphi\in V_c(\Omega\cup\Omega_I)$. We multiply (16) by $P_n(\varphi)\psi$, sum $j=1,...,N$ and integrate with respect to t to discover:

$$\int_0^T\left(\langle u_n^{''},P_n(\varphi)\psi(t)\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(P_n(\varphi)\psi))dydx\right)dt \tag{21}$$
$$=\int_0^T\int_\Omega f_nP_n(\varphi)\psi dxdt$$

by passing to weak limits, together with the fact that $P_n(\varphi)\xrightarrow{V_c(\Omega\cup\Omega_I)}\varphi$ we obtain:

$$\int_0^T\left(\langle u^{''},\varphi\rangle_{V_c^*(\Omega\cup\Omega_I),V_c(\Omega\cup\Omega_I)}\psi(t)+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi\psi(t)))dydx\right)dt \tag{22}$$
$$=\int_0^T\int_\Omega f\varphi\psi(t)dxdt$$

for all $\psi\in L^2(0,T)$ and $\varphi\in V_c(\Omega\cup\Omega_I)$. This terminates the proof.
It remains to prove that $u(0)=g$ and $u^{'}(0)=h$, For this purpose, we choose any function $\varphi\in V_c(\Omega\cup\Omega_I)$ and $\psi\in\mathcal{C}^1([0,T])$ such that $\psi(T)=\psi^{'}(T)=0$. Integrating by parts twice with respect to t in (22) yields:

$$\int_0^T\left(\int_{\Omega\cup\Omega_I}u\varphi\psi^{''}(t)dx+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi\psi(t)))dydx\right)dt \tag{23}$$
$$=\int_0^T\int_\Omega f\varphi\psi(t)dxdt-\int_\Omega u(0)\varphi\psi^{'}(0)dx+\int_\Omega u^{'}(0)\varphi\psi(0)dx$$

Similarly from (21) we get:

$$\int_0^T\left(\int_{\Omega\cup\Omega_I}u_nP_n(\varphi)\psi^{''}(t)dx+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u_n).(\xi.\mathcal{D}^*(P_n(\varphi)\psi(t)))dydx\right)dt \tag{24}$$
$$=\int_0^T\int_\Omega f_nP_n(\varphi)\psi(t)dxdt-\int_\Omega u_n(0)P_n(\varphi)\psi^{'}(0)dx+\int_\Omega u_n^{'}(0)P_n(\varphi)\psi(0)dx$$

by passing to the limit, we obtain:

$$\int_0^T\left(\int_{\Omega\cup\Omega_I}u\varphi\psi^{''}(t)dx+\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}\mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi\psi(t)))dydx\right)dt \tag{25}$$

$$= \int_0^T \int_\Omega f\varphi\psi(t)dxdt - \int_\Omega g\varphi\psi^{'}(0)dx + \int_\Omega h\varphi\psi(0)dx$$

comparing those results, we conclude that $u(0) = g$ and $u^{'} = h$. □

Finally, we announce the uniqueness of the weak solution to (14).

Theorem 3.5. *A weak solution of* (14) *is unique.*

Proof. Since the equation is linear, to show uniqueness it is sufficient to show that the only solution u of (14) with zero data $f = g = h = 0$ is $u = 0$.
To verify this, fix $0 \leq s \leq T$ and set

$$\varphi(t) = \begin{cases} \int_t^s u(\tau)d\tau & \text{if } 0 \leq t < s \\ 0 & \text{if } s \leq t \leq T. \end{cases} \tag{26}$$

Then $\varphi(t) \in V_c(\Omega \cup \Omega_I)$ for each $0 \leq t \leq T$, which allows to write

$$\int_0^s \left(\langle u^{''}, \varphi \rangle_{V_c^*(\Omega\cup\Omega_I), V_c(\Omega\cup\Omega_I)} + \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx \right) dt = 0$$

since $u^{'} = 0$ and $\varphi(s) = 0$ by integrating by parts, we obtain:

$$\int_0^s \left(-\int_{\Omega\cup\Omega_I} u^{'}\varphi^{'}dx + \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(u).(\xi.\mathcal{D}^*(\varphi))dydx \right) dt = 0$$

now as $\varphi^{'} = -u(0 \leq t \leq s)$, we acquire:

$$\int_0^s \left(\int_{\Omega\cup\Omega_I} u^{'}udx - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(\varphi^{'}).(\xi.\mathcal{D}^*(\varphi))dydx \right) dt = 0$$

then

$$\int_0^s \frac{d}{dt} \left(\parallel u \parallel^2_{L^2(\Omega\cup\Omega_I)} - \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} \mathcal{D}^*(\varphi).(\xi.\mathcal{D}^*(\varphi))dydx \right) dt = 0$$

here

$$\parallel u(s) \parallel^2_{L^2(\Omega\cup\Omega_I)} + |||\varphi(0)||| = 0$$

Consequently $u = 0$ on $[0, T]$. □

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