On shakedown and high-cycle fatigue of shape memory alloys in the presence of permanent inelasticity and degradation effects

Michaël Peigney

ON SHAKEDOWN AND HIGH-CYCLE FATIGUE OF SHAPE MEMORY ALLOYS IN THE PRESENCE OF PERMANENT INELASTICITY AND DEGRADATION EFFECTS

MICHAEL PEIGNEY
Université Paris-Est, Laboratoire Navier (UMR 8205), CNRS, Ecole des Ponts ParisTech, IFSTTAR, 77455, Marne la Vallée, France

ABSTRACT

For elastic perfectly plastic bodies under cyclic loadings, Melan's and Koiter's theorems give conditions for the energy dissipation to remain bounded with respect to time. That last situation – classically referred to as shakedown – is associated with the intuitive idea that the body considered behaves elastically in the large time limit. Regarding fatigue design, shakedown corresponds to the most beneficial regime of high-cycle fatigue, as opposed to the regimes of low-cycle fatigue or ratcheting. Although the physical mechanisms in shape memory alloys (SMA) differ from plasticity, the hysteresis observed in the stress-strain response shows that some energy dissipation occurs, and it can be reasonably assumed that situations where the energy dissipation remains bounded is the most favorable regarding the fatigue of SMA bodies. This raises the issue of extending Melan's and Koiter's shakedown theorems to shape memory alloys. In particular, SMA models coupling phase-transformation with permanent inelasticity have been proposed lately to capture degradation effects which are frequently observed experimentally for cyclic loadings. To model such a behavior, two internal variables are generally introduced: in addition to the (constrained) variable describing the phase transformation, an additional variable is used to describe permanent inelasticity. A coupling term between those two variables is generally introduced in the free energy. This communication discusses the extension of Melan's shakedown theorem to such a class of SMA material models. That theorem gives conditions for the energy dissipation to remain bounded, and is expected to be relevant for the fatigue design of SMA structures.

KEYWORDS: SHAKEDOWN, DIRECT METHODS, FATIGUE, SHAPE MEMORY ALLOYS, PERMANENT INELASTICITY

INTRODUCTION

Lately, models coupling phase-transformation and plasticity have been proposed in an effort to describe permanent inelasticity effects which are experimentally observed in SMAs (see e.g. [Auricchio, 2007], [Barrera, 2014], [Gu, 2017]): although phase transformation in SMAs is the main inelastic mechanism, dislocation motions also exist and are (partly) responsible for such effects as training and degradation in cyclic loadings. Extending the approach introduced in [Peigney, 2010] and [Peigney, 2014], we present a static shakedown theorem for SMA models coupling phase-transformation and permanent inelasticity.

CONSTITUTIVE MODEL

In this communication, we consider the constitutive model originally introduced in [Auricchio, 2007] and later refined in [Barrera, 2014]. The state variables are the total strain ε, the transformation strain e^{tr} and the permanent inelastic strain q. Both e^{tr} and q are symmetric trace-free second order tensors. The transformation strain e^{tr} is required to satisfy the constraint:

$$\|e^{tr}\| \leq \varepsilon_L \tag{1}$$

In Eq. (1), ε_L is a material parameter and the norm $\|\cdot\|$is the Euclidean norm.
The Helmholtz free-energy density function is expressed as $\Psi = \Psi_0 + \mathscr{I}_{\varepsilon_L}$ where

$$\begin{aligned}\Psi_0(\varepsilon, e^{tr}, q) = \quad & \tfrac{1}{2}K(\operatorname{tr}\varepsilon)^2 + G\ \|e - e^{tr}\|^2 + \tau_M\|e^{tr} - q\| \\ & + \tfrac{1}{2}H\ \|e^{tr}\|^2 + \tfrac{1}{2}h\|q\|^2 - Ae^{tr} : q\end{aligned} \tag{2}$$

In (2), e is the deviatoric part of ε; K,G, τ_M, H, h, A are all material parameters. The energy term $\mathscr{I}_{\varepsilon_L}(e^{tr})$ is defined by $\mathscr{I}_{\varepsilon_L}(e^{tr}) = 0$ if $\|e^{tr}\| \leq \varepsilon_L$ and $\mathscr{I}_{\varepsilon_L}(e^{tr}) = +\infty$ otherwise. The energy Ψ is strictly convex provided that $hH - A^2 > 0$, which is assumed to be satisfied in the following
The stress-strain relation is obtained by differentiating the free energy function Ψ with respect to the strain ε as:

$$\operatorname{tr}\sigma = 3K\operatorname{tr}\varepsilon, \quad s = 2G(e - e^{tr}) \tag{3}$$

where s is deviatoric part of the stress σ. Since Ψ is only subdifferentiable in (e^{tr}, q), the thermodynamic forces (X, Q) associated to the internal variables (e^{tr}, q) are defined by

$$-(X, Q) \in \partial\Psi \tag{4}$$

where ∂ denotes the subdifferential operator with respect to (e^{tr}, q) [Brézis, 1972].
Let $\mathscr{C}$ be the elasticity domain of the material. A possible choice for $\mathscr{C}$ is $\{(X, Q) : \|X\|^2 + \kappa^2\|Q\|^2 \leq R_Y^2\}$ where κ and R_Y are material parameters (see [Barrera, 2014],

[Peigney, 2018]). A normality flow rule is considerd for the variables (e^{tr}, q):

$$(\dot{e}^{tr}, \dot{q}) = \lambda \left(\frac{X}{\|X\|}, \kappa^2 \frac{Q}{\|Q\|} \right) \quad (5)$$

with conditions $\lambda \geq 0$, $\|X\|^2 + \kappa^2 \|Q\|^2 - R_Y^2 \leq 0$, $\lambda(\|X\|^2 + \kappa^2 \|Q\|^2 - R_Y^2) = 0$.

QUASI-STATIC EVOLUTION OF A CONTINUUM

Consider a continuum occupying a domain Ω and submitted to a prescribed loading history. Body forces are denoted by f^d. Displacements u^d are imposed on a part Γ_u of the boundary Γ, and tractions T^d are prescribed on $\Gamma_T = \Gamma - \Gamma_u$. The given functions f^d,u^d,T^d depend on position x and time t. Assuming quasi-static evolutions, the stress and strain fields satisfy

$$\sigma \in \mathscr{K}_\sigma \ , \ \varepsilon \in \mathscr{K}_\varepsilon \quad (6)$$

where

$$\mathscr{K}_\sigma = \{\sigma : \operatorname{div} \sigma + f^d = 0 \text{ in } \Omega; \sigma.n = T^d \text{ on } \Gamma_T\},$$
$$\mathscr{K}_\varepsilon = \{\varepsilon : \varepsilon = (\nabla u + \nabla^T u)/2 \text{ in } \Omega; u = u^d \text{ on } \Gamma_u\}.$$

Starting from a given initial state, Eqs. (3), (4), (5) and (6) govern the evolution of the continuum. Besides from nonlinearity, the main difficulties in solving the evolution problem lie in the nondifferentiability of the free energy (2) and in the constraint (1) on the internal variable. In the following, we examine conditions under which the energy dissipation remains bounded in time for all solutions of the evolution problem. Such a situation is referred to as shakedown.

SHAKEDOWN THEOREM

Consider the *fictitious elastic response* $(\sigma^E, \varepsilon^E)$ of the continuum, defined by

$$\sigma^E \in \mathscr{K}_\sigma \ , \varepsilon^E \in \mathscr{K}_\varepsilon \ , \ \operatorname{tr} \sigma^E = 3K \operatorname{tr} \varepsilon^E, \quad s^E = 2G e^E$$

where e^E and s^E are the deviatoric part of ε^E and σ^E, respectively. The following result can be proved:

If there exists $m > 1$ and a time-independent field $B(x)$ such that $\|m s^E(x,t) - B(x)\| \leq R_Y$ for all (x,t), then shakedown occurs whatever the initial state.

The proof builds on techniques introduced in [Peigney, 2010] and is not reported here because of space limitations. It can be observed that the obtained shakedown condition reduces to a restriction on the diameter of the curve $t \mapsto s^E(x,t)$. We emphasize that the presented shakedown theorem is path-independent - in the sense that it gives a sufficient condition for shakedown to occur, whatever the initial state is.

CONCLUSION

An interesting feature of the presented shakedown theorem is that it relies only on elastic calculations (so as to obtained the elastic response σ^E). Incremental nonlinear analysis is thus bypassed completely. Since there is an established connection between fatigue and energy dissipation, the proposed theorem is expected to be relevant for the fatigue design of SMA systems in the presence of permanent inelasticity and degradation effects. It should also be mentioned that the presented theorem can be extended to other constitutive models of permanent inelasticity in SMAs. In particular, a nonconvex mixing energy, as notably considered in [Hackl, 2008], [Peigney, 2009], [Peigney, 2013a], [Peigney, 2013b], can be taken into account.

REFERENCES

[Auricchio, 2007] F. Auricchio, A. Reali, and U. Stefanelli, “A three-dimensional model describing stress-induced solid phase transformation with permanent inelasticity,” *Int.J. Plasticity*, 23, 2007, 207–226.

[Barrera, 2014] N. Barrera, P. Biscari, and M. Urbano, “Macroscopic modeling of functional fatigue in shape memory alloys,” *Eur.J.Mech. A.Solids*, 45, 2014, 101–109.

[Brézis, 1972] H. Brézis, “Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de hilbert,” North-Holland, Amsterdam, 1972.

[Gu, 2017] X. Gu, W. Zhang, W. Zaki, and Z. Moumni, “An extended thermomechanically coupled 3d rate-dependent model for pseudoelastic smas under cyclic loading,” *Smart Materials and Structures*, 26, 2017, 095047.

[Hackl, 2008] K. Hackl and R. Heinen, “An upper bound to the free energy of $n-$variant polycrystalline shape memory alloys,” *J.Mech.Phys.Solids*, 56, 2008, 2832–2843.

[Peigney, 2009] M. Peigney, “A non-convex lower bound on the effective free energy of polycrystalline shape memory alloys,” *J.Mech.Phys.Solids*, 57, 2009, 970–986.

[Peigney, 2010] ——,, “Shakedown theorems and asymptotic behaviour of solids in non-smooth mechanics,” *Eur.J.Mech. A.Solids*, 29, 2010, 784–793.

[Peigney, 2013a] ——, “On the energy-minimizing strains in martensitic microstructures-part 1: Geometrically nonlinear theory,” *J.Mech.Phys.Solids*, 61, 2013, 1489–1510.

[Peigney, 2013b] ——, “On the energy-minimizing strains in martensitic microstructures-part 2: Geometrically linear theory,” *J.Mech.Phys.Solids*, 61, 2013 1511–1530.

[Peigney, 2014] ——, “On shakedown of shape memory alloys structures,” *Annals of Solid and Structural Mechanics*, 6, 1-2, 2014, 17–28.

[Peigney, 2018] M. Peigney, G. Scalet, and F. Auricchio, “A time-integration algorithm for a 3d constitutive model for smas including permanent inelasticity and degradation effects,” *International Journal for Numerical Methods in Engineering*, DOI 10.1002/nme.5835, 2018.