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Abstract

The deconvolution problem in image processing consists of reconstructing an original image from an observed and thus a degraded one. This degradation is often modelized as a linear operator plus an additive noise. The linear operator is called the blurring operator and the goal consists of deblurring the image. Very often, the blurring operator is modelized as a convolution whose kernel (the Point Spread Function) is not directly known in practice. In this paper, we first propose a new model for convolution and we validate it through computer simulations. Basically, we expend the kernel leading to a sequence of real coefficients in link with the moment problem. We particularly emphasize the radial isotropic case.

***Index Terms*—** Image restoration, Image deblurring, Blind deblurring, Blind deconvolution, The moment problem.

1. INTRODUCTION

Image deblurring is one of the most discussed problems in image processing since it plays a prominent role in several applied sciences. This problem consists in recovering an original u from a degraded one u_0, by dropping the effects of blur and noise. The connection between u and u_0 is often modelled by the equation

$$u_0 = Ku + n.$$

in which K represents a blur operator, n an additive noise and u_0 is the observed image. In the most common model, K is considered of the form

$$Ku = k \star u, \tag{1}$$

where the function k is called the Point Spread Function (PSF). There exists an abundant literature about the subject, especially concerning non blind deconvolution, that is retrieving the original image u from u_0 when the blur K is known (as for example in denoising problems for which $K = I$). A usual approach in non blind deconvolution consists in solving the minimization problem

$$\min_u \int_\Omega |Ku - u_0|^2 dx + \int_\Omega \theta(|\nabla u|) dx, \tag{2}$$

where θ denotes a suitable function chosen such that interior edges are preserved (see, e. g., [1, 2]). In blind deconvolution problems, both the original image and the blur are unknown and must be recovered from u_0. This is a difficult problem which is often encountered in practical applications such as artistic restoration, medical imaging, astronomical imagery, seismology and some current-life applications decoding bar codes, reading texts using a camera phone (see, e. g., [3, 4, 5]). Supposing that the blur K is described by a parameter p, the following variational model is often used for getting a conjoint estimation of the blur kernel and the shape image

$$\min_{p,u} E(p, u) = \int_\Omega |K(p)u - u_0|^2 dx + \lambda J_1(u) + \mu J_2(k), \tag{3}$$

where J_1 and J_2 are two penalization terms (see [6]). For example, in the case of a radial symmetric out-of-focus blur, the operator K is of the form $Ku = k \star u$ with k given by

$$k(x) = \frac{1}{V_d R^d} \mathbb{1}_{B_r}(x), \tag{4}$$

where $B_r = \{x \in \mathbb{R}^d | |x| < r\}$ and V_d is the volume of the unit ball given by

$$V_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)},$$

where Γ is the well-known gamma function. In this case, r plays the role of the parameter p. One can observe in this example that it is so easy to handle with this parameter $p\ (= r)$ in solving the minimization problem (3). This problem becomes much more difficult when no information on the nature of the PSF k is available. In [7], we propose the first idea, the mathematical framework and some theoretical aspects for a new manner to write the blur operator K which replaces the convolution (1). This form consists to write Ku as sum which approximates the convolution $k \star u$

$$\mathscr{R}(\sigma)u := Ku = \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma_\alpha \partial^\alpha u. \qquad (5)$$

Our propose here is to present some practical issues and our numerical results. Of course in practice, this sum is truncated and the parameter p describing K can be considered as the sequence of the coefficients $(\sigma_\alpha)_\alpha$ appearing in the sum (5). These coefficients are linked to *the moments* of k and could be explicitly computed when k is known. When the blur is supposed centro-symmetric, the expansion (5) writes

$$Ku = \sum_{k \geq 0} \frac{\Gamma(d/2)}{2^{2k}\Gamma(k + d/2)k!} \sigma_k \Delta^k u, \qquad (6)$$

where Δ is the Laplace operator. Notice that in practice, the Laplacian Δ is considered in its discrete form. As we shall show in section 3, from the discrete viewpoint approximations, (5) and (6) converge for any image $u = (u_{i,j})$, provided that some soft conditions are satisfied by the coefficients $(\sigma_\alpha)_\alpha$. In blind deconvolution problems, the parameter p is often unknown and must be computed by solving an inverse problem such as (3). Thus, setting the correct constraints on this parameter is of crucial importance. We shall see that this question is intimately linked to the *moment problem* (see [8, 9, 10] for more details on such problem). The truncature of the sums (5) and (6) and the recovering of the PSF k from the coefficients $(\sigma_\alpha)_\alpha$ are also important questions which will be treated in this paper. The paper is organized as follows : section 2 defines the proposed model and how it approaches the blind deblurring problem. After an introduction of the *moment problem*, we show how to determinate the PSF from the moments. In section 3, we give a practical way to use the sum (6). Finally, some numerical results illustrate this work in Section 4.

2. THE MODEL

We start this section by giving a summarized presentation of the model. In its general version, the model we propose here for the blur can be expressed as follows

- The blur operator K is parametrized by a sequence of positive real numbers $(\sigma_\alpha)_{\alpha \in \mathbb{N}^d}$ and writes into the form
$$Ku = \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma_\alpha \partial^\alpha u, \qquad (7)$$
- The sequence $(\sigma_\alpha)_{\alpha \in \mathbb{N}^d}$ is subject to the following abstract constraint

  there exists a positive function k on $\mathbb{R}^d$ such that
$$\forall \alpha \in \mathbb{N}^d, \int_{\mathbb{R}^d} x^\alpha k(x)dx = \sigma_\alpha. \qquad (8)$$

Condition (8) means that the coefficients $(\sigma_\alpha)_\alpha$ are *the moments* of a non negative function k on $\mathbb{R}^d$. From a practical viewpoint this condition is not tractable in this form and must be made more explicit. We shall see that this question is intimately linked to the well known *moment problem.*
If in addition, we suppose that the blur is radial symmetric (i. e. $k(x) = k(|x|)$), then the approximation (7) becomes

$$Ku = \sum_{k \geq 0} \frac{\Gamma(d/2)}{2^{2k}\Gamma(k + d/2)k!} \sigma_k \Delta^k u, \qquad (9)$$

with the following condition on the coefficients σ_k, $k \geq 0$;
there exists a bounded function ρ, non negative on $\mathbb{R}_+$, such that

$$\sigma_0 = 1,\ \sigma_k = \int_0^{+\infty} t^k \rho(t)dt,\ \text{for } k \geq 0. \qquad (10)$$

This condition can be seen as a one dimensional moment problem on the half-line; it is called the *Stieljes problem*. Notice that the functions ρ and k are linked by the identity

$$k(x) = \frac{2}{A_d}|x|^{2-d}\rho(|x|^2), \qquad (11)$$

where A_d is the surface area of the unit sphere given by

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

More, we can suppose that the function k has a compact support, that is

$$k(x) = 0 \text{ for all } |x| > R,$$

where the parameter R is the radius of the support of k. Setting $\delta = \sqrt{R}$, we can write

$$\sigma_k = \delta^k \sigma_k^\star, \qquad (12)$$

where $(\sigma_k^\star)_{k \geq 0}$ is a sequence of real numbers satisfying the constraint:
there exists a function $\rho^\star$, nonnegative on $[0, 1]$, such that

$$\sigma_0^\star = 1,\ \sigma_k^\star = \int_0^1 t^k \rho^\star(t)dt,\ \text{for } k \geq 0. \qquad (13)$$

Now, we can see that the connection between the model (7) and the convolutive model (9) is evident. The sum (7) approximates formally the convolution $k \star u$, with k solution of (8), that is

$$k \star u \approx \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} \sigma_\alpha \partial^\alpha u.$$

For example, in the case of a radial out-of-focus blur, the PSF k is given by (4) and one has

$$\rho(t) = \frac{d}{2R^d} t^{d/2-1} \mathbb{1}_{[0,r^2]}(t),\ \sigma_k = \frac{d}{d+2k} r^{2k} \text{ for } k \geq 0. \qquad (14)$$

In the case of a gaussian blur, one has

$$k(x) = \frac{1}{(2\pi)^{d/2}\sigma^d}\exp(\frac{-|x|^2}{2\sigma^2}) \quad (15)$$

$$\rho(t) = \frac{1}{2\Gamma(d/2)\sigma^d}\left(\frac{t}{2}\right)^{d/2-1}\exp(-\frac{t}{2\sigma^2}), \quad (16)$$

and

$$\sigma_k = \frac{(2\sigma^2)^k\Gamma(k+d/2)}{\Gamma(d/2)} \text{ for } k \geq 0.$$

In practical problems, we can use expressions (7) or (9) in non blind deconvolution problem, after computing the coefficients $(\sigma_\alpha)_{\alpha\in\mathbb{N}^m}$ or $(\sigma_k)_{k\in\mathbb{N}}$. Indeed, in blind deconvolution problems, the sequence (σ_α) is unknown or partially known (as for the out-of-focus blur) and must be estimated simultaneously with the original image. In other terms, the parameter p can be considered as the sequence (σ_α). In the case of a centro-symmetric and compactly supported PSF, p can be considered as the pair $(\delta, (\sigma_k^\star)_{k\in\mathbb{N}})$. In the simplest case of a radial symmetric out-of-focus blur of the form (4) (resp. a gaussian blur of the form (15)) p is nothing but the unknown radius r (resp. the standard deviation of the gaussian blur σ).

3. CONVERGENCE OF THE EXPANSION AND TRUNCATURE

In this section, we focus our attention on the truncature of expansion (9). By sake of simplicity, we treat only the case of a radial symmetric blur given by the sum (9). Notice that in practice this sum is truncated. Hence, the operator K is replaced by a finite sum

$$\mathscr{R}_N(\sigma)u := K_N(\sigma^\star)u = \sum_{k=0}^{N}\frac{\Gamma(\frac{d}{2})}{2^{2k}\Gamma(k+\frac{d}{2})k!}\sigma_k\Delta^k u, \quad (17)$$

for some integer $N \geq 1$. Two questions pop up in this case; at what order truncate this sum? (b) what are the constraints on the truncated sequence $(\sigma_k)_{k\geq 0}$?.

The purpose of the following section is to give an answer to these questions.

In the discrete form, an image is composed of a set of pixels indexed by (i,j), $1 \leq i \leq N$, $1 \leq j \leq M$. $u = (u_{i,j})_{1\leq i\leq N, 1\leq j\leq M}$ belongs in X, where $X = \mathbb{R}^{N\times M}$. The space X is equipped with the euclidian inner scalar product:

$$\forall u, v \in X,\ \langle u, v\rangle_X = \sum_{i=1}^{N}\sum_{j=1}^{M} u_{i,j}v_{i,j}.$$

By a minor abuse of the notation, we state for X^m, where $m \geq 1$, the space $(\mathbb{R}^m)^{N\times M}$. The gradient of $u \in X$, written ∇u belongs to X^2 and could be defined by several manners. One of them consists to set $\nabla u = (g^{(1)}, g^{(2)})$ with:

$$g^{(1)}_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < N, \\ 0 & \text{if } i = N. \end{cases}$$

$$g^{(2)}_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < M, \\ 0 & \text{if } j = M. \end{cases} \quad (18)$$

The div operator is defined in X^2 to X as the adjoint operator of $-\nabla$. So, for all $p = (p^{(1)}, p^{(2)}) \in X^2$, we have:

$$\forall z \in X,\ \langle \mathrm{div} p, z\rangle = -\langle p, \nabla z\rangle.$$

We state for all $u \in X$,

$$\Delta u = \mathrm{div}(\nabla u). \quad (19)$$

Then, from the definition of the divergence, we have:

$$\forall u, v \in X, \langle \Delta u, v\rangle = -\langle \nabla u, \nabla v\rangle = \langle u, \Delta v\rangle. \quad (20)$$

We set

$$\|\Delta\| = \max_{v\neq 0}\frac{\|\Delta v\|}{\|v\|}.$$

We start with the following lemma

Lemma 1 *Suppose that ρ satisfies the following assumption: there exist four constants $\alpha \geq 0$, $\beta > 1/2$, $\lambda > 0$ and $C \geq 0$ such that*

$$\forall t \in \mathbb{R}_+,\ \rho(t) \leq ct^\alpha\exp(-\lambda t^\beta), \quad (21)$$

Then, the sequence (17) converges normally. Moreover, this normal convergence holds also when $\beta = 1/2$ and $\lambda^2 > \|\Delta\|$.

Condition (21) is satisfied by a Gaussian blur or an out-of-focus blur of the form (4 or 15). It is also satisfied by any compatctly supported kernel.

Remark 1 *With definition (18) of the gradient and divergence operator, one can prove that*

$$8 - 4(\frac{1}{N} + \frac{1}{M}) \leq \|\Delta\|_2 \leq 8.$$

Lemma 2 *Suppose that $n = 2$. Let $(\sigma_k)_{k\geq 0}$ be a real sequence satisfying the condition*

$$0 < \sigma_k \leq M\tau^k, \quad (22)$$

where $\tau > 0$ and $M > 0$ are two real constants. Then, the finite sum (17) converges normally for each $u \in X$. Moreover,

$$\|Ku - K_N u\| \quad \leq \quad \frac{\Gamma(\frac{d}{2})M_N}{2\pi(N+1)}e^{2(N+1)(\ln\theta_N + \theta_N + 1)}\|u\|,$$

where $\theta_N = \dfrac{\tau}{2(N+1)}$ and $M_N = \sup\limits_{k\geq N+1}\sigma_k\tau^{-k} \leq M$.

Proposition 1 *If we suppose (22) and $N + 1 \geq \dfrac{\tau}{2\lambda}$, where λ is the unique real satisfying $\log\lambda + \lambda + 1 = 0$ ($\lambda \approx 0.2785$) , then*

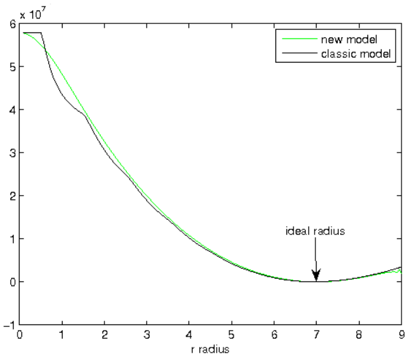
$$\|K(\sigma)u - K_N(\sigma)u\| \leq \frac{\Gamma(\frac{d}{2})}{2\pi}\frac{M_N}{N+1}\|u\|.$$

4. NUMERICAL RESULTS

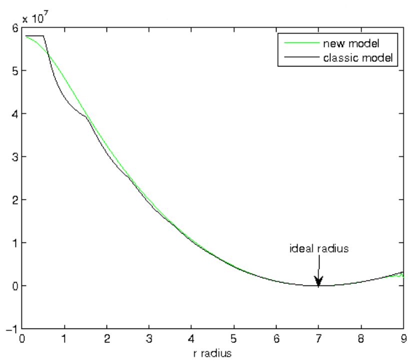
To fix the ideas, on the following tests, we blur an "ideal" image u with an out-of-focus blurring operator k and with the new operator $\mathscr{R}(.)$ defined in (17). Thus, the data (namely the blurry images that we would like to restore) are:

- $u_0 = k * u$,
- $u_0 = \mathscr{R}(\sigma)u$.

The figure 1 compares the criterion evolution with respect to the radius R, between the convolutive model and the proposed one with respect to the criterion (2) with u known. This test shows that the graphs globally coincide and in a neighborhood of the optimum, they are overlaid.



(a) Test with $u_0 = k * u$ and $R = 7$



(b) Test with $u_0 = \mathscr{R}(\sigma)u$ (see 17) and $R = 7$

Fig. 1. Comparison of both criteria with respect to the radius R where u the ideal image is known. 1(a) represents the comparison of the criteria for the data defined by an out-of-focus degraded (blurred) image, the blur radius is equal to $R = 7$ and 1(b) is the comparison of the criteria where the data is defined by the new operator given by (17) for $R = 7$. For both examples, the order of the truncature N = 30.

Now, we suppose that the radius R is known and we would like to find u, an approximation of the ideal image.

The figure 2 presents the criterion (2) in the convolutive case (where k is an out-of-focus kernel (4)) and with proposed

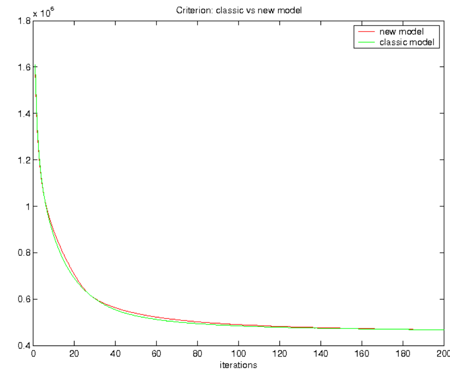
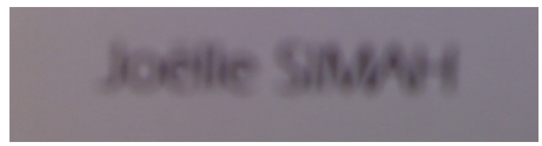
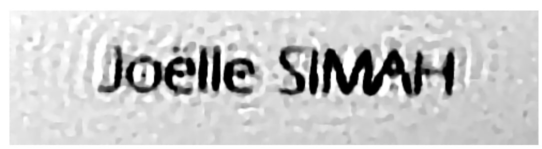


Fig. 2. The criteria decay with respect to the iterations



(a) Original (real) blurry image

(b) restored image by the convolutive model

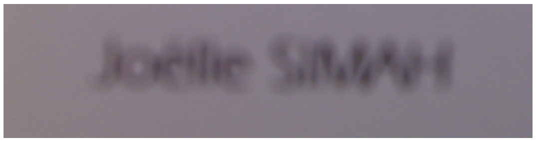


(c) restored image by the new model

Fig. 3. Restoration of an out-of-focus blurred image when R is unknown. 3(a) represents the original image, 3(b) the restored image with the convolutive model and 3(c) the restoration by the proposed model ($R \simeq 10$ with $N = 50$).

model (17) with respect to the iterations. The convergence toward the optimum is quite similar. This shows that, from a numerical viewpoint, the new approach is in accordance with the older approach (convolutive model).

The figure 3 compares the results given by both models. The chosen test (real-life) image has been acquired by a camera phone without autofocus (the mobile : Nokia N70). This image (494×125) is a part of a visit card (V-Card). An interesting application of the proposed method is a preprocessing step in order to scan and recognize text or barcode in document acquired by a camera phone or a digicam (see [4, 5]). Here, of course the blur is unknown, we are in the blind deblurring case. We suppose that the main distortion of this image is an out-of-focus blur plus a gaussian noise. So, we must give an approximation of the blur radius R . Many ap-



(a) original image

(b) binarization of the restored image (convolutive model)

(c) binarization of the restored image (new model)

Fig. 4. Restoration of an out-of-focus blurred image when R is unknown. 4(a) represents the original image, 4(b) a binarization of the restored image with the convolutive model and 4(c) a binarization of the restoration with the proposed one.

proaches may be used, for example we can estimate simultaneously R with the original image or, when we deal with an out-of-focus blur, we can firstly estimate an approximation of R in the cepstral domain (see [11]) then we use our method when R is estimated. An estimation of the radial blur is $R \simeq 10$ and $N = 50$. The figure 4 is a classical binarization of the restored images, it is interesting to notice that the texts (in the restored images) have been read (recognized) by different text recognition software, namely the software "ABBYY" (see [12]) and the software "Cardiris" (see [13]). Of course, they don't recognize the blurred image.

5. CONCLUSION

A novel model for blind deblurring is presented in this paper. Based on the *moment problem* for the PSF estimation, we avoid the convolution which is very expensive. So, we can restore a blurred image in reasonable computation time. In other words, we could expect to obtain a good restoration in fast way (it depends only on a recursive laplacian). Moreover, we propose a robust algorithm allowing a simultaneous computing of the blur kernel and the estimated deblured image. In particular, this approach has been successfully applied to restore blurred images taken from a camera of very poor quality.

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[13] Cardiris : OCR software-and Document-Management-solutions, ," `http://www.irislink.com/`.