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On stress-gradient materials: formulation and homogenization

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Stress-gradient elasticity in a nutshell

The Stress Principle of Cauchy

\[ \boldsymbol{\sigma} \cdot \nabla + b = 0 \quad \text{and} \quad [\boldsymbol{\sigma}] \cdot n = 0 \quad \text{(SA)} \]

Cauchy elasticity

\[ W^c[\sigma] = \int w^c(\sigma) \, dV \]

A mature model

Forest and Sab (2012), Mechanics Research Communications 40 (first derivation)

Sab, Legoll and Forest (2016), Journal of Elasticity 123(2) (mathematical justification)

Forest and Sab (2017), Mathematics and Mechanics of Solids (extension to finite strain)

Alternative model (not discussed here)

Polizzotto (2014, 2016), International Journal of Solids and Structures
In the present talk

A brief overview of the stress-gradient model
  ► Decomposition of the stress-gradient
  ► Stress boundary conditions

Homogenization of stress-gradient materials
  ► Hill–Mandel lemma, boundary conditions
  ► Softening size effect

Application to composites with spherical inclusions
  ► Eshelby’s inhomogeneity problem
  ► Mori–Tanaka estimates of effective compliance

Tran, Brisard, Guillemoinot and Sab (2018), International Journal of Solids and Structures

Some open questions
Derivation of the stress-gradient model for elasticity
Roadmap

Minimize complementary stress energy under SA constraint

- Cauchy elasticity: elastic equilibrium of fixed solid is retrieved
- Stress-gradient elasticity: elastic equilibrium of fixed solid is defined
- At this point, “fixed” not really meaningful, since dofs not (yet) defined
- “Trace” of the stress-gradient is prescribed
Decomposition of stress-gradient (1/2)

Complementary stress energy of stress-gradient materials

\[ W^c[\sigma] = \int w^c(\sigma, \sigma \otimes \nabla) \, dV \]

Classical equilibrium equation

\[(\sigma \otimes \nabla) : I_2 = \sigma \cdot \nabla = -b\]

Orthogonal decomposition of stress gradient

\[ \sigma \otimes \nabla = Q + R \quad \text{with} \quad R : I_2 = 0 \quad \text{and} \quad Q \vdash R = 0 \]

\[ R = I_6' \vdash (\sigma \otimes \nabla) \quad \text{(projection onto space of trace-free, third-rank tensors)} \]

Equivalent expression of complementary stress energy

\[ W^c[\sigma] = \int w^c(\sigma, Q, R) \, dV \]
Decomposition of stress-gradient (2/2)

Orthogonal decomposition of stress gradient

\[ \sigma \otimes \nabla = Q + R \quad \text{with} \quad R : I_2 = 0 \quad \text{and} \quad Q : R = 0 \]

**Q is in fact fully prescribed**

\[ Q = -\frac{1}{2} I_4 \cdot b \]

(straightforward linear algebra)

No strain measure attached to \( Q \) (generalized prestress)

Modeling assumption

\[ W^c[\sigma] = \int w^c(\sigma, Q, R) \, dV = \int w^c(\sigma, R) \, dV \]

with \( R = I'_6 : (\sigma \otimes \nabla) \)
Constrained minimization of energy

Initial problem

Minimize: \( W^c[\sigma] = \int w^c(\sigma, I'_6 \cdot (\sigma \otimes \nabla)) \, dV \)

subject to: \( \sigma \cdot \nabla + b = 0 \)

Stress-gradient as independent variable

Minimize: \( W^c[\sigma, R] = \int w^c(\sigma, R) \, dV \)

subject to: \[
\begin{align*}
\sigma \cdot \nabla + b &= 0 \\
R &= I'_6 \cdot (\sigma \otimes \nabla) \\
\Phi &= u
\end{align*}
\]
Elastic equilibrium of fixed, SG bodies

Field equations

\[ \sigma \cdot \nabla + b = 0 \]
\[ \mathbf{R} = I' \cdot (\sigma \otimes \nabla) \]
\[ \mathbf{e} = \partial_\sigma \mathbf{w}^c \]
\[ \Phi = \partial_\mathbf{R} \mathbf{w}^c \]
\[ \mathbf{e} = \Phi \cdot \nabla + \varepsilon [\mathbf{u}] \]

Generalized equilibrium equations
Generalized stress-strain relations
Generalized strain-displacement relations

Boundary conditions

\[ \Phi \cdot \mathbf{n} + \text{sym} (\mathbf{u} \otimes \mathbf{n}) = 0 \]

6 scalar boundary conditions

Continuity conditions

\[ \left[ \Phi \cdot \mathbf{n} + \text{sym} (\mathbf{u} \otimes \mathbf{n}) \right] = 0 \]
\[ \left[ \sigma \right] = 0 \]

Weaker continuity of “displacements”
Stronger continuity of “stresses”
Linear elasticity

Complementary stress energy density

\[ w^c(\sigma, R) = \frac{1}{2} \sigma : S : \sigma + \frac{1}{2} R : M : R \]

No \( \sigma - R \) coupling for centrosymmetric materials!

\( S \) and \( M \) have major and minor symmetries

Generalized compliance \( M \) operates on trace-free tensors

\[ M = I' \cdot M \cdot I' \]

Stress-strain relationships

\[ e = S : \sigma \quad \text{and} \quad \Phi = M : R \]

or

\[ \sigma = C : e \quad \text{and} \quad R = L : \Phi \]

\[ C = S^{-1} \quad \text{and} \quad L = M^+ \]
Isotropic linear elasticity

General isotropic linear stress-gradient elasticity

\[ M = I'_{6} \quad \therefore \quad M = I'_{6} \]

\[ 2\mu M = \ell_{J}^{2} J_{6} + \ell_{K}^{2} K_{6} + \ell_{H}^{2} H_{6} \]

\[ I'_{6} = J_{6} + K_{6} \]

Simplified model for isotropic linear stress-gradient elasticity

\[ 2\mu S = \frac{1-2\nu}{1+\nu} J_{4} + K_{4} \]

\[ \frac{2\mu}{\ell^{2}} M = \frac{1-2\nu}{1+\nu} J_{6} + K_{6} \]

Altan and Aifantis (1992), Scripta Metallurgica et Materialia 26(2)
Altan and Aifantis (1997), Journal of the Mechanical behavior of Materials 8(3)
Homogenization
Setting the stage for homogenization

The 3 cases

- Stress-gradient (micro) → stress-gradient (macro)
- Cauchy (micro) → stress-gradient (macro)
- Stress-gradient (micro) → Cauchy (macro)

Separation of scales

- Classical condition
  
  \[ d \ll L_{\text{meso}} \ll L_{\text{macro}} \]

- Additional condition for stress-/strain- gradient materials
  
  \[ \ell \sim d \quad \text{or} \quad \ell \ll d \]
Effective and apparent properties

Effective behaviour: stress-gradient → Cauchy  \( (\ell \sim d) \)

\[ \langle e \rangle = S^{\text{eff}} : \langle \sigma \rangle \]

Apparent elastic properties of (large) SVE \( \Omega \)

\[ \langle e \rangle = S^{\text{app}}(\Omega) : \langle \sigma \rangle \]

\[ \langle e \rangle = \frac{1}{V} \int_{\Omega} \sigma \, dV \]

\[ \langle e \rangle = \frac{1}{V} \int_{\Omega} \epsilon[u] \, dV + \frac{1}{V} \int_{\partial\Omega} \Phi \cdot n \, dS \]

Local problem on SVE \( \Omega \)

\[ \sigma \cdot \nabla = 0 \]

\[ e = S : \sigma \quad \Phi = M : (\sigma \otimes \nabla) \]

\[ e = \Phi \cdot \nabla + \epsilon[u] \]

No body forces!

+ boundary conditions!
Boundary conditions

Generalized Hill–Mandel lemma

\[ \langle \sigma^* : e + (\sigma^* \otimes \nabla) : \Phi \rangle = \langle \sigma^* \rangle : \langle e \rangle \]

Uniform stress boundary conditions

\[ \sigma \big|_{\partial \Omega} = \overline{\sigma} \]

\[ \langle e \rangle = S^\sigma(\Omega) : \langle \sigma \rangle = S^\sigma(\Omega) : \overline{\sigma} \]

Uniform strain boundary conditions

\[ \Phi \cdot n + \text{sym}(u \otimes n) = \text{sym}[(\overline{e} \cdot x) \otimes n] \]

\[ \langle \sigma \rangle = C^\varepsilon(\Omega) : \langle e \rangle = C^\varepsilon(\Omega) : \overline{e} \]

Uniform traction boundary conditions

\[ \sigma \big|_{\partial \Omega} \cdot n = \overline{\sigma} \cdot n \]

\[ \langle e \rangle = S^T(\Omega) : \langle \sigma \rangle = S^T(\Omega) : \overline{\sigma} \]

\[ a \cdot [\Phi \cdot n + \text{sym}(u \otimes n)] \cdot a = 0 \]

\[ a = l_2 - n \otimes n \]
Variational formulation

Minimum of complementary stress energy

$$\overline{\sigma}: S^\sigma(\Omega): \overline{\sigma} = \inf \{ \langle \sigma : S : \sigma + (\sigma \otimes \nabla) : M : (\sigma \otimes \nabla) \rangle \}$$

subject to: $$\sigma \cdot \nabla = 0$$ and $$\sigma|_{\partial \Omega} = \overline{\sigma}$$

Minimum of strain energy

$$\overline{\varepsilon}: C^\varepsilon(\Omega): \overline{\varepsilon} = \inf \{ \langle \varepsilon[u] : C : \varepsilon[u] + \Phi : L : \Phi \rangle \}$$

subject to: $$\Phi \cdot n + \text{sym}(u \otimes n) = \text{sym}[(\overline{\varepsilon} \cdot x) \otimes n]$$

Bounds for finite-size SVEs

$$C^\sigma(\Omega_I) \leq C^\sigma(\Omega_{II}) \leq C^{\text{eff}} \leq C^\varepsilon(\Omega_{II}) \leq C^\varepsilon(\Omega_I) \text{ for } \Omega_I \subset \Omega_{II}$$

Huet (1990), Journal of the Mechanics and Physics of Solids 38(6)
Softening size-effect

Variational definition of apparent compliance (uniform $\sigma$ BC)

If $S_I \leq S_{II}$ and $M_I \leq M_{II}$ everywhere in $\Omega$, then $S_{I}^{\text{eff}} \leq S_{II}^{\text{eff}}$

Fixed microstructure, variable material internal length

For $S_I = S_{II}$ and $M_I = \frac{\ell_I^2}{\ell_{II}^2} M_{II}$: if $\ell_I \leq \ell_{II}$ then $S_{I}^{\text{eff}} \leq S_{II}^{\text{eff}}$

More generally: if $\left(\frac{\ell}{d}\right)_I \leq \left(\frac{\ell}{d}\right)_{II}$ then $S_{I}^{\text{eff}} \leq S_{II}^{\text{eff}}$

Scaled microstructure, constant material internal length

If $d_I \leq d_{II}$ then $C_{I}^{\text{eff}} \leq C_{II}^{\text{eff}}$  

Softening size-effect!
Eshelby’s inhomogeneity problem for stress-gradient elasticity
Problem setting

Matrix: $\mu_m, \nu_m, \ell_m$

$\sigma^\infty e_z$

$-\sigma^\infty e_z$

$\nu_i = \nu_m = 0.25$

$\mu_i = 10 \mu_m$
Some tedious calcs later…

Hat tip to VP Tran!

http://www.sympy.org

Axial stress along vertical axis

Axial stress at origin

Theorem of Eshelby (1957) does not hold!

Dilute stress concentration tensor

\[ \langle \sigma \rangle_i = \frac{1}{V_i} \int_{\Omega_i} \sigma \, dV = B^\infty : \sigma^\infty \]

Axial stress along vertical axis

Axial stress at origin

\[ \ell_{m/a} = 0.1, 0.5, 1.0 \]
Mori–Tanaka estimates for composites with spherical inclusions
Problem setting

\[ \mu_i = 10 \mu_m, \nu_i = 0.25, \ell_i \]

\[ \mu_m, \nu_m = 0.25, \ell_m \]

Volume fraction of inclusions: \( f \)
Estimates of bulk and shear moduli

Strain- and stress-gradient models are not equivalent!

Effective bulk modulus

$$S_{\text{eff}} = S_m + f \left( S_i - S_m \right) : B^\infty : \left[ (1-f)I_4 + f B^\infty \right]^{-1}$$

Benveniste (1987), Mechanics of Materials 6(2)
Ma and Gao (2014), Acta Mechanica 225(4-5)
Some questions
The total complementary energy

Assumption 1: Stress Principle of Cauchy

\[ \sigma \cdot \nabla = 0 \quad \text{and} \quad \sigma \cdot n \big|_{\partial \Omega} = \overline{T} \]

Assumption 2: complementary work of prescribed displacements

\[ V^c = \int_{\partial \Omega_u} \overline{u} \cdot \sigma \cdot n \, dS \]

Minimization of complementary energy

Minimize: \[ \Pi^c = W^c - V^c \]

subject to: \[ \sigma \cdot \nabla = 0 \quad \text{and} \quad \sigma \cdot n \big|_{\partial \Omega} = \overline{T} \]
The general BVP

Field equations

\[ \sigma \cdot \nabla + b = 0 \]
\[ R = I' \cdot (\sigma \otimes \nabla) \]
\[ e = \partial_\sigma w^c \]
\[ \Phi = \partial_R w^c \]
\[ e = \Phi \cdot \nabla + \varepsilon [u] \]

Boundary conditions

\[ \partial \Omega_T : \left\{ \begin{array}{l}
\sigma \cdot n |_{\partial \Omega} = \overline{T} \\
a \cdot [\Phi \cdot n + \text{sym}(u \otimes n)] \cdot a = 0
\end{array} \right. \]
\[ a = l_2 - n \otimes n \]
\[ \partial \Omega_u : \Phi \cdot n + \text{sym}(u \otimes n) = \text{sym}(\overline{u} \otimes n) \]

Continuity conditions

\[ [\sigma] = 0 \]
\[ [\Phi \cdot n + \text{sym}(u \otimes n)] = 0 \]
The total potential energy

Potential strain energy

\[ W = \int w(e, \Phi) \, dV \]

Work of prescribed body forces and tractions

\[ V = \int b \cdot u \, dV + \int_{\partial \Omega_T} \bar{T} \cdot [u + 2 \Phi \cdot n \otimes n - (\Phi : n \otimes n \otimes n) n] \, dS \]

Minimization of total potential energy

Minimize: \[ \Pi = W - V \]

subject to:

\[ \begin{align*}
\partial \Omega_T & : \ a \cdot [\Phi \cdot n + \text{sym}(u \otimes n)] \cdot a = 0 \\
\partial \Omega_u & : \ \Phi \cdot n + \text{sym}(u \otimes n) = \text{sym}(\bar{u} \otimes n)
\end{align*} \]
Conclusion & perspectives

Summary

► Simplified model for isotropic linear elasticity
► General framework for stress-gradient → Cauchy homogenization
► Softening size-effect
► Mori–Tanaka estimates for spherical inclusions

Outlook

► Hashin–Shtrikman (size-dependent) bounds on effective properties
► Cauchy → stress-gradient homogenization
► Understand the meaning of $u$ (use above)
Thanks for your attention!

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