Equivalence Between Time Consistency and Nested Formula

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May 17, 2019

Abstract

Figure out a situation where, at the beginning of every week, one has to rank every pair of stochastic processes starting from that week up to the horizon. Suppose that two processes are equal at the beginning of the week. The ranking procedure is time consistent if the ranking does not change between this week and the next one. In this paper, we propose a minimalist definition of Time Consistency (TC) between two (assessment) mappings. With very few assumptions, we are able to prove an equivalence between Time Consistency and a Nested Formula (NF) between the two mappings. Thus, in a sense, two assessments are consistent if and only if one is factored into the other. We review the literature and observe that the various definitions of TC (or of NF) are special cases of ours, as they always include additional assumptions. By stripping off these additional assumptions, we present an overview of the literature where the specific contributions of authors are enlightened. Moreover, we present two classes of mappings, translation invariant mappings and Fenchel-Moreau conjugates, that display time consistency under suitable assumptions.

Keywords: Dynamic Risk Measure, Time Consistency, Nested Formula

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1 Introduction

Behind the words "Time Consistency" and "Nested Formula", one can find a vast literature resorting to economics, dynamical risk measures and stochastic optimization.

Let us start with economics. In a dynamic bargaining problem, a group of agents has to agree on a common path of actions. As time goes on and information is progressively revealed, they can all reconsider the past agreement, and possibly make new assessments leading to new actions. Stability is the property that the agents will stick to their previous commitment. Time consistency is a form of stability when an individual makes a deal between his different selves (agents) along time. The notion of "consistent course of action" (see Peleg and Yaari, 1973) is well-known in the field of economics, with the seminal work of Strotz (1955-1956): an individual having planned his consumption trajectory is consistent if, reevaluating his plans later on, he does not deviate from the originally chosen plan. This idea of consistency as "sticking to one's plan" may be extended to the uncertain case where plans are replaced by decision rules ("Do thus-and-thus if you find yourself in this portion of state space with this amount of time left", Richard Bellman cited in Dreyfus (2002)); Hammond (1976) addresses "consistency" and "coherent dynamic choice", Kreps and Porteus (1978) refers to "temporal consistency". Another classical reference in economics is Epstein and Schneider (2003).

Dynamic or Time Consistency has been introduced in the context of dynamical risk measures (see Riedel, 2004; Detlefsen and Scandolo, 2005; Cheridito et al., 2006; Artzner et al., 2007, for definitions and properties of coherent and consistent dynamic risk measures).

In the field of stochastic optimization, Time Consistency has then been studied for Markov Decision Processes by Ruszczyński (2010).

These different origins of Time Consistency contribute to a disparate literature. First, as Nested Formulas lead naturally to Time Consistency, some authors study the conditions to obtain Nested Formulas, whereas others focus on the axiomatics of Time Consistency and obtain Nested Formulas. Second, many definitions cohabit. For instance, Ruszczyński (2010) add translation invariant property with additive criterion, Shapiro (2016); Artzner, Delbaen, Eber, Heath, and Ku (2007) add assumptions of coherent risk measures, and many authors focus on a particular structure of information (filtration). In this disconnected landscape, De Lara and Leclère (2016) tries to make the connection between "dynamic consistency" for optimal control problems (economics, stochastic optimization) and "time consistency" for dynamic risk measures. In this paper, we will focus on Time Consistency, motivated by dynamic risk measures — where the future assessment of a tail of a process is consistent with the initial assessment of the whole process, head and tail — but not limited to them. Below, we sketch our definitions of TC and NF. Our main contribution will be proving their equivalence. Let $\mathbb{H}$ and $\mathbb{T}$ be two sets, respectively called *head set* and *tail set*. Let $\mathbb{A}$, $\mathbb{F}$ be two sets and let A and F be two mappings as follows:

$$A : \mathbb{H} \times \mathbb{T} \to \mathbb{A} \;, \quad F : \mathbb{T} \to \mathbb{F} \;. \tag{1}$$

The mapping A is called an *aggregator*, as it aggregates head-tail in $\mathbb{H} \times \mathbb{T}$ into an element of $\mathbb{A}$. The mapping F is called a *factor* because of the Nested Formula (NF).

Axiomatic for Time Consistency. We start presenting axiomatic of Time Consistency in a nutshell. Depending on the authors, the objects that are manipulated are either processes (Riedel, 2004; Detlefsen and Scandolo, 2005; Cheridito, Delbaen, and Kupper, 2006; Artzner, Delbaen, Eber, Heath, and Ku, 2007) or lotteries (Kreps and Porteus, 1978; Epstein and Schneider, 2003). These objects are divided into two parts: a head h and a tail t. On the one hand, we have a way to assess any tail t by means of a mapping F (factor), yielding $F(t)$. On the other hand, we have a way to assess any couple head-tail (h, t) by means of a mapping A (aggregator), yielding $A(h, t)$.

We look for a consistency property between these two ranking mappings F and A: if a tail t is equivalent to a tail t', then the two elements (h, t) and (h, t') — that share the same head — must be such that (h, t) is equivalent to (h, t'). This can be written mathematically as

$$F(t) = F(t') \Rightarrow A(h, t) = A(h, t') \ , \ \ \forall (h, t, t') \in \mathbb{H} \times \mathbb{T}^2 \ . \tag{TC}$$

Axiomatic for Nested Formulas. Some authors focus on sufficient conditions to obtain a Nested Formula (Shapiro, 2016; Ruszczynski and Shapiro, 2006). In a Nested Formula, the assessment $F(t)$ of any tail t is factored inside the assessment $A(h, t)$ of any head-tail (h, t) by means of a surrogate mapping $S^{A,F}$ as follows:

$$A(h, t) = S^{A,F}\big(h, F(t)\big) \ . \tag{NF}$$

Of course, (NF) implies (TC). We will prove the reverse: (TC) implies that there exists a mapping $S^{A,F}$ such that (NF) holds true.

In Sect. 2, we go through the literature, with the goal of extracting the following components: what kind of objects are treated, what are the heads and the tails, how these objects are ranked. In Sect. 3, we formally state our definitions of Time Consistency (TC) and Nested Formula (NF), and we prove their equivalence. We also provide conditions to obtain analytical properties of the mapping $S^{A,F}$ appearing in the Nested Formula, such as monotonicity, continuity, convexity, positive homogeneity and translation invariance. In Sect. 4, we show that our framework covers the different frameworks reviewed in Sect. 2. Finally, in Sect. 5, we present two classes of mappings, translation invariant mappings and Fenchel-Moreau conjugates, that display time consistency under suitable assumptions.

2 Review of the literature

We have screened a selection of papers, in mathematics and economics, touching Time Consistency and Nested Formula in various settings. Depending on the setting, we identify the following components, as introduced in Sect. 1: what kind of objects are treated, what are the heads and the tails, how are these objects ranked. Table 1 sums up our survey.

	Article	Objects	Head	Tail	Assessment
Time Consistency	Kreps and Porteus (1978)	Lottery	Lottery from 1 to s	Lottery from $s+1$ to T	Expected utility
	Epstein and Schneider (2003)	Lottery	Lottery from 1 to s	Lottery from $s+1$ to T	Not necessarily expected utility
	Ruszczyński (2010)	Process	Process from 1 to s	Process from $s+1$ to T	Dynamic risk measure
	Artzner, Delbaen, Eber, Heath, and Ku (2007)	Process	Process from 1 to τ	Process from τ to T, τ stopping time	Coherent risk measure
Nested Formula	Shapiro (2016)	Process	Process from 1 to s	Process from $s+1$ to T	Coherent risk measure
	Ruszczynski and Shapiro (2006)	Process	Process from 1 to s	Process from $s+1$ to T	Coherent risk measure
	De Lara and Leclère (2016)	Process	Process from 1 to s	Process from $s+1$ to T	Dynamic risk measure

Table 1: Sketch of papers selected on Time Consistency and Nested Formulas

2.1 Axiomatic for Time Consistency (TC)

The first group of authors is subdivided between economists, who deal with lotteries and preferences, and probabilists who deal with stochastic processes and dynamical risk measures.

2.1.1 Lotteries and preferences

In Kreps and Porteus (1978), Kreps and Porteus (1979) and Epstein and Schneider (2003), the authors deal with lotteries and preferences. A preference is a total, transitive and reflexive relation. Proper assumptions make it possible that the preference relation can be represented by a numerical evaluation. Assumptions of monotonicity and convexity are also made.

In Kreps and Porteus (1978), the authors propose axioms that make that the preference is represented by an expected utility formula.

By contrast, more general numerical representations are studied in Epstein and Schneider (2003), even if the authors add an hypothesis of additive criterion. A summary of the assumptions can be found in Table 2.

2.1.2 Dynamic risk measures and processes

In Ruszczyński (2010) and Artzner, Delbaen, Eber, Heath, and Ku (2007), the authors deal with stochastic processes assessed by dynamical risk measures.

In Ruszczyński (2010), the author studies a family of conditional risk measures which are monotonic, invariant by translation and homogeneous. The criterion is additive.

In Artzner, Delbaen, Eber, Heath, and Ku (2007), the authors focus on the value of the stochastic process at the final time step. They use as assessment a particular class of risk measures, the so-called coherent risk measures.

2.2 Axiomatic for Nested Formulas (NF)

In Shapiro (2016), Ruszczynski and Shapiro (2006) and De Lara and Leclère (2016), the focus is on exhibiting sufficient conditions to obtain Nested Formulas. All authors study stochastic processes, with an assumption of monotonicity for the assessment, but there are some differences.

In Ruszczynski and Shapiro (2006), the authors study coherent risk measures in their dual form (hence with properties of convexity, invariance by translation and additive criterion).

In Shapiro (2016), the author focuses on assessing the value of the process at the final step with coherent risk measures.

In De Lara and Leclère (2016), the author study how commutation properties between time aggregators and uncertainty aggregators make it possible to obtain Nested Formulas.

	Article	Monotonicity	Translation invariance	Convexity
Time Consistency	Kreps and Porteus (1978)	Yes	No	Yes
	Kreps and Porteus (1979)	Yes	No	Yes
	Epstein and Schneider (2003)	Yes	No	Yes
	Ruszczyński (2010)	Yes	Yes	No
	Artzner, Delbaen, Eber, Heath, and Ku (2007)	Yes	Yes	Yes
Nested Formula	Shapiro (2016)	Yes	Yes	Yes
	Ruszczynski and Shapiro (2006)	Yes	Yes	Yes
	De Lara and Leclère (2016)	Yes	No	No

Table 2: Most common assumptions in the selection of papers on Time Consistency and Nested Formula

3 Main result: equivalence between time consistency and nested formula

In Sect. 1, we have sketched the notions of Time Consistency and Nested Formula. Now, in §3.1, we properly define Weak Time Consistency — with minimal assumptions — and we prove that it is equivalent to a Nested Formula. In §3.2, we extend definitions and results to Usual and Strong Time Consistency: by adding order structures, we obtain additional properties. In §3.3, we provide conditions to obtain analytical properties of the mapping appearing in the Nested Formula, such as monotonicity, continuity, convexity, positive homogeneity and translation invariance. Let us introduce basic notations.

Let $\mathbb{H}$ and $\mathbb{T}$ be two sets, respectively called *head set* and *tail set*. Let $\mathbb{A}$, $\mathbb{F}$ be two sets and let A and F be two mappings as follows:

$$A : \mathbb{H} \times \mathbb{T} \to \mathbb{A} \; , \quad F : \mathbb{T} \to \mathbb{F} \; . \tag{2}$$

The mapping A is called an *aggregator*, as it aggregates head-tail in $\mathbb{H} \times \mathbb{T}$ into an element of $\mathbb{A}$. The mapping F is called a *factor* because of the Nested Formula (NF) in Sect. 1.

Definition 3.1. With the couple aggregator-factor (A, F) in (2) we associate the set-valued mapping

$$\begin{aligned} S^{A,F} : \mathbb{H} \times \mathrm{Im}(F) &\rightrightarrows \mathbb{A} \\ (h, f) &\mapsto S^{A,F}(h, f) = \big\{ A(h, t) \mid t \in F^{-1}(f) \big\} , \end{aligned} \tag{3}$$

where $\mathrm{Im}(F) = F(\mathbb{T})$. We call $S^{A,F}$ the *subaggregator* of the couple (A, F).

3.1 Weak Time Consistency

Definition 3.2 (Weak Time Consistency)**.** The couple aggregator-factor (A, F) in (2) is said to satisfy *Weak Time Consistency (WTC)* if we have

$$F(t) = F(t') \Rightarrow A(h, t) = A(h, t') \ , \ \ \forall h \in \mathbb{H} \ , \ \ \forall (t, t') \in \mathbb{T}^2 \ . \tag{4}$$

Here is our main result where we characterize the WTC property in terms of the subaggregator in (3).

Theorem 3.3 (Nested decomposition of WTC mappings)**.** *The couple aggregator-factor (A, F) in (2) is WTC if and only if the subaggregator set valued mapping $S^{A,F}$ in (3) is a mapping. In that case, the following* Nested Formula *between mappings holds true:*

$$A(h, t) = S^{A,F}\big(h, F(t)\big) \ , \ \ \forall h \in \mathbb{H} \ , \ \ \forall t \in \mathbb{T} \ . \tag{5}$$

Proof. Note that we always have by Equation (3) that

$$A(h, t) \in S^{A,F}\big(h, F(t)\big) \ . \tag{6}$$

1. We suppose that the couple (A, F) is Weak Time Consistent. Consider (h, f) fixed in $\mathbb{H} \times \mathrm{Im}(F)$. We are going to show that the set valued mapping $S^{A,F}$ is in fact a mapping, by proving that the set $S^{A,F}(h, f)$, defined in (3), is reduced to a singleton. We consider two elements $a = A(h, t)$ and $a' = A(h, t')$ in the set $S^{A,F}(h, f)$. By definition (3), we have $F(t) = F(t') = f$. Then, using the Weak Time Consistency property (4), we deduce $A(h, t) = A(h, t')$. Thus, $S^{A,F}(h, f)$ is reduced to one value for $f \in \mathrm{Im}(F)$. The set valued mapping $S^{A,F}$ is thus a mapping and, using Equation (6), we obtain $A(h, t) = S^{A,F}\big(h, F(t)\big)$.

2. We suppose now that the set valued mapping $S^{A,F}$, defined in (3), is a mapping. Since $S^{A,F}$ is a mapping, we deduce by Equation (6) that $A(h, t) = S^{A,F}\big(h, F(t)\big)$ for all $t \in \mathbb{T}$. Therefore, we have the implications: $F(t) = F(t') \Rightarrow S^{A,F}\big(h, F(t)\big) = S^{A,F}\big(h, F(t')\big) \Rightarrow A(h, t) = A(h, t')$. We conclude that the weak time consistency property (4) is satisfied.

In both cases, we have shown that Equation (5) holds true. □

Example 3.4 (The couple $(\mathrm{AV@R}_\beta[\cdot+\cdot]\,,\ \mathrm{AV@R}_\beta[\cdot \mid \mathcal{F}])$ is not Weak Time Consistent)**.** We now give an example inspired from (Pflug and Pichler, 2014, Sect. 5.3.2, p. 188) and involving the well known Average Value at Risk. It helps to illustrate our main result and the notions we have introduced so far.

Let $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, that we equip with the uniform probability distribution $\mathbb{P} = \frac{1}{4}\delta_{\omega_1} + \frac{1}{4}\delta_{\omega_2} + \frac{1}{4}\delta_{\omega_3} + \frac{1}{4}\delta_{\omega_4}$.

We introduce the sets $\mathbb{H} = \mathbb{T} = \mathbb{R}^{|\Omega|} = \mathbb{R}^4$. On this finite space Ω, the Average Value at Risk of level β $(0 \le \beta \le 1)$ of a random variable $\boldsymbol{X} : \Omega \to \mathbb{R}$ is defined by Rockafellar and Uryasev (2000)

$$\mathrm{AV@R}_\beta(\boldsymbol{X}) = \min_{\alpha\in\mathbb{R}} \quad \Big\{ \alpha + \frac{1}{1-\beta}\mathbb{E}_{\mathbb{P}}\big[[\boldsymbol{X}-\alpha]^+\big] \Big\} \ . \tag{7}$$

Let $\mathcal{F} = \big\{\emptyset, \{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}, \Omega\big\}$ be a σ-field on the space Ω. The Conditional Average Value at Risk of level β, of a random variable $\boldsymbol{X} : \Omega \to \mathbb{R}$ with respect to the σ-field $\mathcal{F}$ is defined by (Ruszczyński (2010), Example 3):

$$\mathrm{AV@R}_\beta(\boldsymbol{X} \mid \mathcal{F}) = \inf_{\boldsymbol{U} \preceq \mathcal{F}} \Big\{ \boldsymbol{U} + \frac{1}{1-\beta}\mathbb{E}_{\mathbb{P}}\big[[\boldsymbol{X}-\boldsymbol{U}]^+ \mid \mathcal{F}\big] \Big\} \ , \tag{8}$$

where the infimum is understood point-wise among all random variables $\boldsymbol{U}$ that are $\mathcal{F}$-measurable, and where the level β may be an $\mathcal{F}$-measurable function with values in an interval $[\beta_{\min}, \beta_{\max}] \subset [0,1)$.

We define two mappings

$$A : \mathbb{H}\times\mathbb{T} \to \mathbb{R} \qquad\qquad F : \mathbb{T} \to \mathbb{R}^2 \tag{9a}$$

$$(h,t) \mapsto \mathrm{AV@R}_{0.5}[h+t] \ , \qquad\qquad t \mapsto \mathrm{AV@R}_{0.5}[t \mid \mathcal{F}] \ . \tag{9b}$$

Consider four elements: a head $h_0 = (0,0,0,0) \in \mathbb{H}$, a first tail $t_0 = (3,3,2,1) \in \mathbb{T}$, a second tail $t_0' = (1,3,2,2) \in \mathbb{T}$ and an element of the factor's image $f_0 = (3,2) \in \mathbb{F}$. On the one hand, the elements $F(t_0)$ and $F(t_0')$ are equal, because

$$\mathrm{AV@R}_{0.5}[t_0|\mathcal{F}] = \underbrace{(3;2)}_{f_0} = \mathrm{AV@R}_{0.5}[t_0'|\mathcal{F}] \ . \tag{10}$$

On the other hand, the elements $A(h_0,t_0)$ and $A(h_0,t_0')$ are not equal, because

$$3 = \mathrm{AV@R}_{0.5}[h_0+t_0] \neq \mathrm{AV@R}_{0.5}[h_0+t_0'] = 2.5 \ . \tag{11}$$

The subaggregator $S^{A,F}$ in (3) is not a mapping since

$$S^{A,F}(h_0,f_0) = \big\{\mathrm{AV@R}_{0.5}[h_0+t] \mid \mathrm{AV@R}_{0.5}[t \mid \mathcal{F}] = f_0\big\} \supset \{2.5;\ 3\} \ , \tag{12}$$

and therefore the couple (A,F) in (9) is not Weak Time Consistent.

3.2 Extensions to Usual and Strong Time Consistency

With additional order structures on the image sets $\mathbb{A}$ and $\mathbb{F}$ of the aggregator A and of the factor F, and possibly on the head set $\mathbb{H}$ — all presented in (2) — we define two additional notions of Time Consistency, usual and strong.

3.2.1 Usual Time Consistency (UTC)

Suppose that the image sets $\mathbb{A}$ and $\mathbb{F}$ are equipped with orders, denoted by $\leq$.

Definition 3.5 (Definition of Usual Time Consitency)**.** The couple aggregator-factor (A, F) in (2) is said to satisfy *Usual Time Consistency (UTC)* if we have

$$F(t) \leq F(t') \Rightarrow A(h,t) \leq A(h,t') \ , \ \ \forall h \in \mathbb{H} \ , \ \ \forall (t,t') \in \mathbb{T}^2 \ . \tag{13}$$

We extend the result of Theorem 3.3 as follows.

Proposition 3.6 (Nested decomposition of UTC mappings)**.** *The couple* (A, F) *in* (2) *is UTC if and only if the set valued mapping* $S^{A,F}$ *in* (3) *is a mapping and is* increasing[1] in its second argument. *In that case, the Nested Formula* (5) *holds true.*

The proof is left to the reader as it follows the proof of Theorem 3.3 with small variations.

3.2.2 Strong Time Consistency (STC)

Suppose that the head set $\mathbb{H}$ and the image sets $\mathbb{A}$ and $\mathbb{F}$ are equipped with orders, denoted by $\leq$.

Definition 3.7 (Definition of Strong Time Consistency)**.** The couple (A, F) in Equation (2) is said to satisfy *Strong Time Consistency* (STC) if we have

$$\left. \begin{array}{rcl} F(t) & \leq & F(t') \\ h & \leq & h' \end{array} \right\} \Rightarrow A(h,t) \leq A(h',t') \ , \ \ \forall (h,h',t,t') \in \mathbb{H}^2 \times \mathbb{T}^2 \ . \tag{14}$$

We extend the results of Theorem 3.3 as follows.

Proposition 3.8 (Nested decomposition for STC mappings)**.** *The couple* (A, F) *in* (2) *is STC if and only if the set valued mapping* $S^{A,F}$ *is a mapping* increasing in its first and second arguments. *In that case, the Nested Formula* (5) *holds true.*

The proof is left to the reader as it follows the proof of Theorem 3.3 with small variations.

3.2.3 Summing up results about WTC, UTC and STC

In §3.1 and §3.2, we have introduced three notions of Time Consistency, from the weakest to the strongest. Of course, we have that a Strong Time Consistent couple is also Usual Time Consistent, and that a Usual Time Consistent couple is also Weak Time Consistent. We sum up the different definitions and results in Table 3.

[1] Let $\mathbb{X}$ and $\mathbb{Y}$ be sets endowed with orders denoted by $\leq$. A mapping $M : \mathbb{X} \to \mathbb{Y}$ is said to be *increasing* if $x \leq x' \Rightarrow M(x) \leq M(x')$.

	Weak (4) $\Leftarrow$	Usual (13) $\Leftarrow$	Strong (14)
Definition	$F(t) = F(t')$ $\Downarrow$ $A(h,t) = A(h,t')$	$F(t) \leq F(t')$ $\Downarrow$ $A(h,t) \leq A(h,t')$	$h \leq h'$, $F(t) \leq F(t')$ $\Downarrow$ $A(h,t) \leq A(h',t')$
Characterization in terms of subaggregator	$S^{A,F}$ is a mapping	$S^{A,F}$ is a mapping increasing in its second argument	$S^{A,F}$ is a mapping increasing in both arguments

Table 3: Characterization of Time Consistency in terms of subaggregator

3.3 Analytical properties of time consistent mappings

Here, we study properties inherited by the subaggregator $S^{A,F}$ in (3) when it is a mapping, that is, when the couple (A, F) is Weak Time Consistent (see Theorem 3.3). We insist that, in this part, we study how properties of the subaggregator $S^{A,F}$ can be deduced from properties of aggregator A and factor F. Thus, our approach differs from other approaches in the literature, like Ruszczynski and Shapiro (2006), where properties of A are deduced from properties of $S^{A,F}$ and F. We focus on monotonicity, continuity, convexity, positive homogeneity and translation invariance.

3.3.1 Monotonicity

We suppose that the head set $\mathbb{H}$, the tail set $\mathbb{T}$, and the image sets $\mathbb{A}$ and $\mathbb{F}$ — all presented in (2) — are equipped with orders, denoted by $\leq$. The proof of the following proposition is left to the reader as a direct application of the Nested Formula (5).

Proposition 3.9 (Monotonicity)**.** *Let the couple (A, F) be Weak Time Consistent, as in Definition 3.2. If the mapping A is increasing in its first argument, then the subaggregator $S^{A,F}$ in (3) is increasing in its first argument.*

3.3.2 Continuity

We suppose that the head set $\mathbb{H}$, the tail set $\mathbb{T}$, and the image sets $\mathbb{A}$ and $\mathbb{F}$ are metric spaces.

Proposition 3.10 (Continuity)**.** *Let the couple (A, F) be Weak Time Consistent, as in Definition 3.2. Assume that the tail set $\mathbb{T}$ is compact. If the factor F is continuous and if the aggregator A is continuous with a compact image $\mathrm{Im}(A) = A(\mathbb{H} \times \mathbb{T})$, then the subaggregator $S^{A,F}$ in (3) is continuous on $\mathbb{H} \times \mathrm{Im}(F)$.*

Proof. We prove the continuity of the subaggregator $S^{A,F}$ on $\mathbb{H} \times \mathrm{Im}(F)$ by using the sequential characterization of the continuity on metric spaces. For this purpose, we consider, on the one hand, $(\bar{h}, \bar{f})$ element of $\mathbb{H} \times \mathrm{Im}(F)$ and, on the other hand, $(h_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathbb{H}$ converging to $\bar{h}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathrm{Im}(F)$ converging to $\bar{f}$. We will show that $S^{A,F}(h_n, f_n)$ converges to

$S^{A,F}(\bar{h},\bar{f})$. We introduce the notation $\mathcal{L}\big(\{u_n\}\big)$ to denote the set of limit points of a sequence $(u_n)_{n\in\mathbb{N}}$.

As $f_n \in \mathrm{Im}(F)$, there exists an element $t_n \in \mathbb{T}$ such that $F(t_n) = f_n$ for each n. By the Nested Formula (5), we deduce that

$$A(h_n,t_n) = S^{A,F}\big(h_n,F(t_n)\big) = S^{A,F}\big(h_n,f_n\big) \; . \tag{15}$$

We will now show that the set $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big)$ of limit points is reduced to the singleton $\{S^{A,F}(\bar{h},\bar{f})\}$. The proof is in several steps as follows:

1. $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) \neq \emptyset$,
2. $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) \subset A\big(\bar{h},\mathcal{L}(\{t_n\})\big)$,
3. $A\big(\bar{h},\mathcal{L}(\{t_n\})\big)$ is reduced to the singleton $\{S^{A,F}(\bar{h},\bar{f})\}$,
4. $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) = \{S^{A,F}(\bar{h},\bar{f})\}$.

Here is the proof.

1. As the sequence $\big(A(h_n,t_n)\big)_{n\in\mathbb{N}}$ takes value in the compact set $\mathrm{Im}(A)$, we have that $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) \neq \varnothing$.
2. We prove that $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) \subset A\big(\bar{h},\mathcal{L}(\{t_n\})\big)$. Let a be an element of the set $\mathcal{L}\big(\{A(h_n,t_n)\}\big)$. By definition of this latter set, there exists a subsequence $\big(A(h_{\Phi(n)},t_{\Phi(n)})\big)_{n\in\mathbb{N}}$ converging to a. Now, we know that $(h_{\Phi(n)})_{n\in\mathbb{N}}$ converges to $\bar{h}$, but it is not necessarily the case that $(t_{\Phi(n)})_{n\in\mathbb{N}}$ converges. However, by compacity of the tail set $\mathbb{T}$, there exist a subsequence $(t_{\Psi\circ\Phi(n)})_{n\in\mathbb{N}}$ converging to a certain $\bar{t} \in \mathcal{L}\big(\{t_n\}\big)$. As the sequence $\big(A(h_{\Phi(n)},t_{\Phi(n)})\big)_{n\in\mathbb{N}}$ is converging to a, the subsequence $\big(A(h_{\Psi\circ\Phi(n)},t_{\Psi\circ\Phi(n)})\big)_{n\in\mathbb{N}}$ is also converging to a. Now that both inner subsequences converge, we use the continuity of the mapping A, and obtain that $a = \lim_{n\to\infty} A(h_{\Psi\circ\Phi(n)},t_{\Psi\circ\Phi(n)}) = A(\bar{h},\bar{t}) \in A\big(\bar{h},\mathcal{L}(\{t_n\})\big)$.
3. We prove the equality $A\big(\bar{h},\mathcal{L}(\{t_n\})\big) = \{S^{A,F}(\bar{h},\bar{f})\}$. Since the set $\mathcal{L}\big(\{t_n\}\big)$ is not empty by compactness of $\mathbb{T}$, we consider $(\bar{t},\bar{t}') \in \mathcal{L}\big(\{t_n\}\big)^2$ any two limits points of the sequence $(t_n)_{n\in\mathbb{N}}$. As $F(t_n) = f_n$ and $\lim_{n\to\infty} f_n = \bar{f}$, we deduce that $F(\bar{t}) = \bar{f} = F(\bar{t}')$, by continuity of the factor mapping F. The Nested Formula (5) gives

   $$A(\bar{h},\bar{t}) = S^{A,F}\big(\bar{h},F(\bar{t})\big) = S^{A,F}(\bar{h},\bar{f}) = S^{A,F}\big(\bar{h},F(\bar{t}')\big) = A(\bar{h},\bar{t}') \; .$$

   This proves that $A\big(\bar{h},\mathcal{L}(\{t_n\})\big) = \{S^{A,F}(\bar{h},\bar{f})\}$.
4. Gathering up the previous results, we obtain that

   $$\varnothing \neq \mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) \subset A\big(\bar{h},\mathcal{L}(\{t_n\})\big) = \{S^{A,F}(\bar{h},\bar{f})\} \; . \tag{16}$$

   We conclude that $\mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) = \{S^{A,F}(\bar{h},\bar{f})\}$.

From Equation (15), we have the equalities $\mathcal{L}\big(\big\{S^{A,F}\big(h_n,f_n\big)\big\}\big) = \mathcal{L}\big(\big\{A(h_n,t_n)\big\}\big) = \{S^{A,F}(\bar{h},\bar{f})\}$. Therefore, the sequence $S^{A,F}(h_n,f_n)$ converges to $S^{A,F}(\bar{h},\bar{f})$. This ends the proof. ∎

3.3.3 Convexity

As we are dealing with convexity property, we assume that the sets $\mathbb{H}, \mathbb{T}$ and $\mathbb{F}$ in (2) are vector spaces. We also suppose that the aggregator $A : \mathbb{H} \times \mathbb{T} \to \mathbb{A}$ in (2) takes extended real values, that is, $\mathbb{A} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Proposition 3.11. *Let the couple (A, F) be Weak Time Consistent, as in Definition 3.2. If there exists a nonempty convex subset $\bar{\mathbb{T}} \subset \mathbb{T}$ such that $F(\bar{\mathbb{T}}) = \mathrm{Im}(F)$ and that the restricted function $F_{|\bar{\mathbb{T}}}$ is affine, and if the aggregator A is jointly convex, then the subaggregator $S^{A,F}$ in (3) is jointly convex on $\mathbb{H} \times \mathrm{Im}(F)$.*

Before entering the proof, let us stress the point that, even if the assumption that the restricted function $F_{|\bar{\mathbb{T}}}$ be affine may look strong, it is quite realistic and widespread. Indeed, for example, if the factor mapping F is the identity mapping on $\mathbb{F}$, then it satisfies the conditions of Proposition 3.11: Conditional expectation or Conditional Average Value at Risk are hence encompassed in this framework.

Proof. We introduce the notation $\mathrm{epi}(M)$ to denote the epigraph [2] of a mapping M. We prove that the subaggregator $S^{A,F}$ is jointly convex by showing that its epigraph is jointly convex.

Let $\big((h_1, f_1), a_1\big)$ and $\big((h_2, f_2), a_2\big)$ be two elements of the epigraph $\mathrm{epi}(S^{A,F})$ of the subaggregator. We consequently have $a_1 \geq S^{A,F}(h_1, f_1)$ and $a_2 \geq S^{A,F}(h_2, f_2)$ which by addition to

$$\lambda a_1 + (1-\lambda)a_2 \geq \lambda S^{A,F}(h_1, f_1) + (1-\lambda)S^{A,F}(h_2, f_2) \; , \tag{17}$$

where λ is an element of $[0, 1]$. As, by assumption, $F(\bar{\mathbb{T}}) = \mathrm{Im}(F)$, there exist two elements $(\bar{t}_1, \bar{t}_2) \in \bar{\mathbb{T}}^2$ such that

$$F(\bar{t}_1) = f_1 \text{ and } F(\bar{t}_2) = f_2 \; . \tag{18}$$

We have the succession of equalities and inequality

$$\begin{aligned}
\lambda a_1 + (1-\lambda)a_2 &\geq \lambda S^{A,F}(h_1, f_1) + (1-\lambda)S^{A,F}(h_2, f_2) \; , && \text{(by Eq. (17),)} \\
&= \lambda S^{A,F}\big(h_1, F(\bar{t}_1)\big) + (1-\lambda)S^{A,F}\big(h_2, F(\bar{t}_2)\big) \; , && \text{(by Eq. (18),)} \\
&= \lambda A(h_1, \bar{t}_1) + (1-\lambda)A(h_2, \bar{t}_2) \; , && \text{(by Eq. (5),)} \\
&\geq A\big(\lambda h_1 + (1-\lambda)h_2, \lambda\bar{t}_1 + (1-\lambda)\bar{t}_2\big) \; , && \text{(by convexity of } A\text{,)} \\
&= S^{A,F}\Big(\lambda h_1 + (1-\lambda)h_2, F\big(\lambda\bar{t}_1 + (1-\lambda)\bar{t}_2\big)\Big) \; , && \text{(by Eq. (5),)} \\
&= S^{A,F}\big(\lambda h_1 + (1-\lambda)h_2, \lambda F(\bar{t}_1) + (1-\lambda)F(\bar{t}_2)\big) \; , \\
& && \text{(by affinity of } F \text{ on } \bar{\mathbb{T}}\text{,)} \\
&= S^{A,F}\big(\lambda h_1 + (1-\lambda)h_2, \lambda f_1 + (1-\lambda)f_2\big) \; , && \text{(by Eq. (18).)}
\end{aligned}$$

We deduce that the element $\Big(\big(\lambda h_1 + (1-\lambda)h_2, \lambda f_1 + (1-\lambda)f_2\big), \lambda a_1 + (1-\lambda)a_2\Big)$ is in the epigraph $\mathrm{epi}(S^{A,F})$ of the subaggregator. This ends the proof. □

[2] Let $\mathbb{X}$ be a set. The epigraph of the mapping $M : \mathbb{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $\mathrm{epi}(M) = \big\{(x, y) \in \mathbb{X} \times \mathbb{R} : M(x) \leq y\big\}$ where y is a real number.

Notice that, if the factor F is only convex, we cannot conclude in general. For example, let $A(h,t) = h + t$ be an aggregator and let $F(t) = \exp(t)$ be a factor. Then the couple (A, F) is Weak Time Consistent with an associated subaggregator $S^{A,F}(h, f) = h + \ln(f)$ which is not convex.

3.3.4 Homogeneity

As we are dealing with homogeneity property, we assume that the sets $\mathbb{H}, \mathbb{T}, \mathbb{A}$ and $\mathbb{F}$ in (2) are endowed with an external multiplication with the scalar field $\mathbb{R}$.

Proposition 3.12 (Positive homogeneity). *Let the couple (A, F) be Weak Time Consistent, as in Definition 3.2. If the mapping A is jointly positively homogeneous and if the mapping F is positively homogeneous, then the subaggregator $S^{A,F}$ is jointly positively homogeneous.*

Proof. Let (h, t) be element of $\mathbb{H} \times \mathbb{T}$. Let $\lambda \in \mathbb{R}^+$. We have the following equalities

$$\begin{aligned} S^{A,F}\big(\lambda h, \lambda F(t)\big) &= S^{A,F}\big(\lambda h, F(\lambda t)\big) \ , && \text{(by positive homogeneity of } F\text{)} \\ &= A(\lambda h, \lambda t) \ , && \text{(by the Nested Formula (5))} \\ &= \lambda A(h, t) \ , && \text{(by positive homogeneity of } A\text{)} \\ &= \lambda S^{A,F}\big(h, F(t)\big) \ , && \text{(by the Nested Formula (5).)} \end{aligned}$$

This ends the proof. □

3.3.5 Translation invariance

As we are dealing with translation invariance, we assume that the sets $\mathbb{H}, \mathbb{T}, \mathbb{A}$ and $\mathbb{F}$ in (2) are endowed with an addition $+$. We also assume that there exists a set $\mathbb{I}$ of invariants which is a common subspace of $\mathbb{H}, \mathbb{T}, \mathbb{A}$ and $\mathbb{F}$, as follows.

Definition 3.13. Let $\mathbb{X}$ and $\mathbb{Y}$ be sets equipped with an addition $+$. Let $\mathbb{I} \subset \mathbb{X} \cap \mathbb{Y}$ be a common subset of $\mathbb{X}$ and $\mathbb{Y}$. A mapping $M : \mathbb{X} \to \mathbb{Y}$ is said to be $\mathbb{I}$-*translation invariant* if

$$M(x + i) = M(x) + i \ , \quad \forall x \in \mathbb{X} \ , \quad \forall i \in \mathbb{I} \ . \tag{19}$$

Proposition 3.14. *Let the couple (A, F) be Weak Time Consistent, as in Definition 3.2. If the mapping A is jointly translation invariant and if the mapping F is translation invariant then the subaggregator $S^{A,F}$ is jointly translation invariant.*

Proof. Let (h, t) be an element of $\mathbb{H} \times \mathbb{T}$. Let $i \in \mathbb{I}$. We have the following equalities:

$$\begin{aligned} S^{A,F}\big(h + i, F(t) + i\big) &= S^{A,F}\big(h + i, F(t + i)\big) \ , && \text{(by translation invariance of } F\text{)} \\ &= A(h + i, t + i) \ , && \text{(by the Nested Formula (5))} \\ &= A(h, t) + i \ , && \text{(by translation invariance of } A\text{)} \\ &= S^{A,F}\big(h, F(t)\big) + i \ , && \text{(by the Nested Formula (5).)} \end{aligned}$$

We conclude that the subaggregator $S^{A,F}$ is jointly translation invariant. □

4 Revisiting the literature

In Sect. 2, we have gone through a selection of papers, touching Time Consistency and Nested Formula in various settings. In Sect. 3, we have formally stated our (abstract) definitions of Time Consistency (TC) and Nested Formula (NF), and we have proven their equivalence. We have also provided conditions to obtain analytical properties of the mapping $S^{A,F}$ appearing in the Nested Formula, such as monotonicity, continuity, convexity, positive homogeneity and translation invariance.

Now, we return to the literature that we have briefly reviewed in Sect. 2, and we show how our framework applies. For this purpose, we go through each article and try to answer two questions.

First, what are the core assumptions that relate to our minimal notions of Time Consistency or Nested Formula? In particular, what are the heads and the tails and how are the Time Consistency axiom or the Nested Formula formulated? We will recover that the various definitions in the selection appear as special cases of ours.

Second, what are the assumptions that are additional to the core TC or NF formulations, and what do they imply for the subaggregator in the Nested Formula? We will extract the additional assumptions specific to each author and hence highlight their additional contribution.

4.1 Axiomatic for Time Consistency (TC)

We start our survey with the group of authors stating Time Consistency axiomatic. This group is subdivided between economists, who deal with lotteries and preferences, and probabilists, who deal with stochastic processes and dynamical risk measures.

4.1.1 Lotteries and preferences

Kreps and Porteus (Kreps and Porteus (1978), Kreps and Porteus (1979)) state a temporal consistency axiom (Axiom 2.1) in the first paper. In the second paper, they focus on the particular case of two stage problems. Their axiomatic is an instance of our Definition 3.5 of Usual Time Consistency. With our Proposition 3.6, we directly deduce the existence of a subaggregator increasing in its second argument and a Nested Formula, whereas they obtain a stronger result under stronger assumptions. Indeed, they add assumptions of continuity, substitution (related to convexity) and focus on Usual Time Consistency with strict inequalities. This enables them to obtain a subaggregator which is continuous and strictly increasing in its second argument and is defined by ((Lemma 4, Theorem 2) and Proposition 1 respectively): $u_{y_t} : \left\{(z, \gamma) \in Z_t \times \mathbb{R} : \gamma = U_{y_t,z}(x) \text{ for some } x \in \mathbb{X}_{t+1}\right\} \to \mathbb{R}$.

Epstein and Schneider Epstein and Schneider (2003) state an axiom of Dynamic Consistency (Axiom 4: DC) which is a particular case of our Definition 3.5 of Usual Time Consistency. With our Proposition 3.6, we directly deduce the existence of a subaggregator increasing in its second argument and a Nested Formula, whereas they obtain a stronger result under stronger assumptions. Indeed, they introduce four additional axioms — Conditional Preferences (CP), Multiple Priors

(MP), Risk Preference (RP) and Full Support (FS) — that ensure a particular form of the subaggregator. MP and CP ensure that the subaggregator can be represented as a minimum of expectation over a rectangular set of probabilities which is closed and convex. MP and RP ensure that the criterion is additive over time. FS ensures that the probability measures have full support. Epstein and Schneider obtain the following Nested Formula[3] associated to Time Consistency (Theorem 3.2): $V_t(h,\omega) = \min_{m \in \mathcal{P}_t^{+1}(\omega)} \int \Big[u\big(h_t(\omega)\big) + \beta V_{t+1}(h) \Big] dm$.

4.1.2 Dynamic risk measures and processes

Ruszczyńsky studies Ruszczyński (2010) dynamic risk measures $\{\rho_{s,T}\}_{s=1}^{T}$. Time Consistency (his Definition 3), appears as a particular case of our Usual Time Consistency Definition 3.5. With our Proposition 3.6, we directly deduce the existence of a subaggregator increasing in its second argument and a Nested Formula, whereas Ruszczyńsky obtains a stronger result under stronger assumptions. Indeed, he adds assumptions that induce a particular form for the subaggregator. From a conditional risk measure $\rho_{s,T}$, he defines mappings $\rho_{s,s'}$ with $s \leq s' \leq T$. With our notations for aggregator A and factor F, he then focuses on the case where the initial assessment is $A = \rho_{s,T}$ and the future assessment is $F = \rho_{s',T}$. With two additional assumptions of invariance by translation and normalization ($\rho_{s,T}(0) = 0$), Ruszczyńsky is able to state that the subaggregator has the specific form (Theorem 1): $S^{A,F} = \rho_{s,s'}$.

In Artzner, Delbaen, Eber, Heath, and Ku (2007), Artzner, Delbean, Eber, Heath and Ku present Time Consistency (their Definition 4.1) which appears as an instance of our Definition 3.5 of Usual Time Consistency. With our Proposition 3.6, we directly deduce the existence of a subaggregator increasing in its second argument and a Nested Formula, whereas they obtain a stronger result under stronger assumptions. Indeed, they study particular mappings of the form $\Psi_t = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot \mid \mathcal{F}_t]$, where $\mathcal{P}$ is a subset of probabilities and $(\mathcal{F}_t)_{t=0}^{T}$ is a filtration. They make an intermediary step before presenting a Nested Formula. They use a tool that they name stability by pasting (rectangularity) of the set $\mathcal{P}$ of probability distributions. With our notations for aggregator A and factor F, this enables them to obtain, for $s \leq s'$, that if $A = \Psi_s$ and $F = \Psi_{s'}$ then the subaggregator has the specific form (Theorem 4.2): $S^{A,F}(h,\cdot) = \Psi_s(h + \cdot)$.

4.2 Axiomatic for Nested formulas (NF)

Shapiro and Ruszczyński Ruszczynski and Shapiro (2006) study a family of conditional risk mapping $\rho_t = \rho_{\mathcal{X}_2|\mathcal{X}_1} \circ \cdots \circ \rho_{\mathcal{X}_t|\mathcal{X}_{t-1}}$ (Equation (5.8)). Each ρ_t is increasing and is associated with a σ-algebra $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t=2}^{T}$ is a filtration. As these mappings ρ_t are instances of the mappings in our Nested Formula (5), they are Usual Time Consistent, by using our Proposition 3.6. With our notations for aggregator A and factor F, and with additional assumptions of monotonicity, translation invariance, convexity and homogeneity, Shapiro and Ruszczyński obtain that, if the initial

[3] The equation is the original transcription of the formula in Epstein and Schneider (2003), to which we refer the reader for a better understanding. By laying it out, we only want to stress the Nested Formula between V_t and V_{t+1}.

assessment is $A = \rho_t$ and the future assessment is $F = \rho_{t+1}$, then the subaggregator is (Theorem 5.1) $S^{A,F} = \rho_{\mathcal{X}_{t+1}|\mathcal{X}_t}$.

Shapiro (Shapiro (2016)) focuses on a future assessment and on a subaggregator of the form (Definition 2.1) $F = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[\cdots \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot \mid \mathcal{F}_{T-1}] \bigm| \mathcal{F}_0\big]$, $S^{A,F} = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot]$. With our notations for aggregator A and factor F, this Nested Formula is an instance of our Nested Formula (5). We can define a natural initial assessment which is Usual Time Consistent with the future assessment, by using our Proposition 3.6. With additional assumptions of finiteness, Shapiro obtains that there exists a bounded set $\widehat{\mathcal{P}}$ of probability distributions such that the initial assessment has the specific form (Theorem 2.1) $A = \sup_{\mathbb{P}\in\widehat{\mathcal{P}}} \mathbb{E}_{\mathbb{P}}[\cdot]$. Besides, with additional assumption (Theorem 2.2) that $\mathcal{P}$ is convex, bounded and weakly closed, Shapiro establishes that $\mathcal{P} = \hat{\mathcal{P}}$.

De Lara and Leclère De Lara and Leclère (2016) study composition of one time step aggregators. They make a distinction between uncertainty aggregator and time step aggregator, and they write a Nested Formula (Equation (11)) which is an instance of our Formula (5). We can naturally define an initial assessment from this composition operation which is time consistent with the one time step aggregator, by using our Proposition 3.6. They add an additional hypothesis of monotonicity and one of commutation between uncertainty aggregator and time aggregator. They deduce that the initial assessment can be defined as the composition between a one time step aggregator (subaggregator) and a future assessment (Theorem 9).

5 Two classes of time consistent mappings

In this section, we present two classes of mappings that display time consistency under suitable assumptions. We study in Sect. 5.1 translation invariant mappings motivated by the representation of risk measures in terms of acceptance set. Then, in Sect. 5.2, we study mappings that are defined as Fenchel-Moreau conjugates motivated by the dual reformulation of convex risk measures.

5.1 Time consistent translation invariant mappings

We study translation invariant mappings defined on ordered groups. We associate to each such mapping an acceptance set which is the level set of level 0. We prove that time consistency between two translation invariant mappings is equivalent to an inclusion between acceptance sets.

5.1.1 Translation invariant mappings on a group

We provide here the definition of a translation invariant mapping and the one of an acceptance set. With these notions, we will state our contribution. We first recall the definition of an ordered group.

Definition 5.1. The triplet $(\mathbb{F}, \oplus, \leq)$ is said to be an *ordered group* if $\mathbb{F}$ is a set, $(\mathbb{F}, \oplus)$ is a group, $(\mathbb{F}, \leq)$ is an ordered set, and the order $\leq$ is compatible with $\oplus$, i.e.

$$f_1 \leq f_2 \Rightarrow f_1 \oplus f_3 \leq f_2 \oplus f_3 \ , \quad \forall (f_1, f_2, f_3) \in \mathbb{F}^3 \ . \tag{20}$$

We now provide the definition of translation invariant mappings on a group.

Definition 5.2. Let $(\mathbb{T}, \oplus)$ be a commutative group and $(\mathbb{F}, \oplus)$ be a subgroup of $(\mathbb{T}, \oplus)$, that we denote by

$$(\mathbb{F}, \oplus) \subset (\mathbb{T}, \oplus) \ . \tag{21}$$

A $(\mathbb{T}, \mathbb{F})$-*translation invariant mapping* is a mapping $F : \mathbb{T} \to \mathbb{F}$ that satisfies

$$F(t \oplus f) = F(t) \oplus f \ , \quad \forall t \in \mathbb{T} \ , \quad \forall f \in \mathbb{F} \ . \tag{22}$$

In addition, if $(F, \oplus, \leq)$ is an ordered group, we introduce the notations $\mathcal{A}_F$ and $\mathcal{A}_{F|\mathbb{F}}$ to deal with particular level sets of the $(\mathbb{T}, \mathbb{F})$-translation invariant mapping $F : \mathbb{T} \to \mathbb{F}$:

$$\mathcal{A}_F = \big\{ t \in \mathbb{T} \mid F(t) \leq 0 \big\} \ , \tag{23a}$$

$$\mathcal{A}_{F_{|\mathbb{F}}} = \big\{ f \in \mathbb{F} \mid F(f) \leq 0 \big\} = \mathcal{A}_F \cap \mathbb{F} \ . \tag{23b}$$

5.1.2 Characterization of UTC in terms of acceptance sets

Given two translation invariant mappings F and ρ as in Definition 5.2, we will build an aggregator A_ρ such that the couple (A_ρ, F) is time consistent as in Definition 3.5.

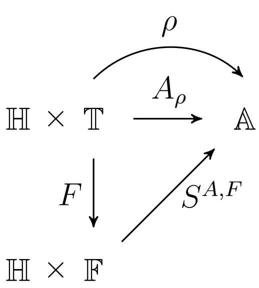

Figure 1: Representation of links between mappings of Proposition 5.3

The next proposition is a generalization, with our notations, of Lemma 11.14 and Proposition 11.15 of Föllmer and Schied (2016), since we do not refer to risk measures on probability spaces but to more general sets.

Proposition 5.3. *Let* $(\mathbb{T}, \oplus)$ *be a commutative group. Given two subgroups*

$$(\mathbb{H}, \oplus) \subset (\mathbb{T}, \oplus) \ \textit{and} \ (\mathbb{A}, \oplus) \subset (\mathbb{T}, \oplus) \ , \tag{24}$$

and a $(\mathbb{T}, \mathbb{A})$*-translation invariant mapping* $\rho : \mathbb{T} \to \mathbb{A}$*, we define the mapping* $A_\rho : \mathbb{H} \times \mathbb{T} \to \mathbb{A}$ *by*

$$A_\rho : \mathbb{H} \times \mathbb{T} \to (\mathbb{A}, \oplus, \leq) \ , \tag{25}$$

$$(h, t) \mapsto \rho(h \oplus t) \ . \tag{26}$$

Let $F : \mathbb{T} \to (\mathbb{F}, \oplus, \leq)$ *be a* $(\mathbb{T}, \mathbb{F})$*-translation invariant mapping. If we have that*

- $(\mathbb{H}, \oplus) \subset (\mathbb{A}, \oplus) \subset (\mathbb{F}, \oplus) \subset (\mathbb{T}, \oplus)$,
- *the* $(\mathbb{T}, \mathbb{A})$*-translation invariant mapping* $\rho : \mathbb{T} \to \mathbb{A}$ *is increasing,*
- *the* $(\mathbb{T}, \mathbb{F})$*-translation invariant mapping* $F : \mathbb{T} \to \mathbb{F}$ *satisfies* $F(0) = 0$ *(where* 0 *is the neutral element of* $(\mathbb{T}, \oplus)$*),*

then the couple of mappings (A_ρ, F) *is Usual Time Consistent if and only if*

$$\mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} = \mathcal{A}_\rho \ , \tag{27}$$

where $\mathcal{A}_F$*,* $\mathcal{A}_{\rho|\mathbb{F}}$ *and* $\mathcal{A}_\rho$ *are defined in* (23a) *and* (23b)*.*

Proof. We refer the reader to Appendix A for the proof. □

Equation (27) establishes a nice relation between acceptance sets of the original mapping ρ and the "conditional" mapping F. However, it remains difficult to solve when the variables are the mappings $\rho : \mathbb{T} \to \mathbb{A}$ and $F : \mathbb{T} \to \mathbb{F}$ given in Proposition 5.3 since it is an implicit equation in ρ.

5.2 Time consistent convex mappings

Here, we focus on time consistency for mappings that are defined as Fenchel-Moreau conjugates. We are motivated by results on dual representation of convex risk mappings (Artzner, Delbaen, Eber, and Heath, 1999).

We first recall Fenchel-Moreau conjugacy with general couplings (not necessarily the classic duality pairing). Then, we state our main theorem that provide a nested formula and hence time consistency of mappings defined as Fenchel-Moreau conjugates.

5.2.1 Basic tools to deal with Fenchel-Moreau conjugacies

The formal tools of couplings and Fenchel-Moreau conjugates were introduced in the seminar paper of Moreau (1970). We recall that $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$.

When we manipulate functions with values in $\overline{\mathbb{R}}$, we adopt the Moreau *lower addition* or *upper addition* defined in Equations (28a) and (28b), depending on whether we deal with sup or inf operations. We only recall useful definitions to make the article self-contained. In the sequel, u, v and w are any elements of $\bar{\mathbb{R}}$.

The Moreau *lower addition* and *upper addition* extend the usual addition with

$$(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty \ , \tag{28a}$$

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty \ . \tag{28b}$$

and they display the following properties:

$$-(u \dot{+} v) = (-u) \underset{\cdot}{+} (-v) \ , \ \ -(u \underset{\cdot}{+} v) = (-u) \dot{+} (-v) \ . \tag{29a}$$

$$\sup_{a \in \mathbb{A}} f(a) \underset{\cdot}{+} \sup_{b \in \mathbb{B}} g(b) = \sup_{a \in \mathbb{A}, b \in \mathbb{B}} \big(f(a) \underset{\cdot}{+} g(b) \big) \ , \tag{29b}$$

Background on Fenchel-Moreau conjugacy with respect to a coupling. Let be given two sets $\mathbb{C}$ and $\mathbb{C}^\sharp$. Consider a *coupling* function $\Phi : \mathbb{C} \times \mathbb{C}^\sharp \to [-\infty, +\infty]$. We also use the notation $\mathbb{C} \overset{\Phi}{\leftrightarrow} \mathbb{C}^\sharp$ for a coupling, so that

$$\mathbb{C} \overset{\Phi}{\leftrightarrow} \mathbb{C}^\sharp \iff \Phi : \mathbb{C} \times \mathbb{C}^\sharp \to [-\infty, +\infty] \; . \tag{30}$$

Definition 5.4. The *Fenchel-Moreau conjugate* of a function $f : \mathbb{C} \to [-\infty, +\infty]$, with respect to the coupling Φ in (30), is the function $f^\Phi : \mathbb{C}^\sharp \to [-\infty, +\infty]$ defined by

$$f^\Phi(c^\sharp) = \sup_{c \in \mathbb{C}} \Big(\Phi(c, c^\sharp) \dot{+} \big(- f(c) \big) \Big) \; , \quad \forall c^\sharp \in \mathbb{C}^\sharp \; . \tag{31}$$

5.2.2 Main result: nested formula for Fenchel-Moreau conjugates

We provide a nested formula between mappings defined as Fenchel-Moreau conjugates. We introduce the notion of decomposable coupling.

Definition 5.5. Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{Y}'$ be four sets and let $\theta_{\mathbb{X}\times\mathbb{Z}}$, $\theta_{\mathbb{Z}}$ and $\theta_{\mathbb{X}}$ be three mappings with values in $\mathbb{Y}'$

$$\theta_{\mathbb{X}\times\mathbb{Z}} : \mathbb{X} \times \mathbb{Z} \to \mathbb{Y}' \; , \tag{32a}$$
$$\theta_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Y}' \; , \tag{32b}$$
$$\theta_{\mathbb{X}} : \mathbb{X} \to \mathbb{Y}' \; . \tag{32c}$$

Let $\varphi : \mathbb{Y}' \times \mathbb{Y} \to [-\infty, +\infty]$ be a coupling between $\mathbb{Y}'$ and $\mathbb{Y}$.

We say that the coupling φ is $(\theta_{\mathbb{X}\times\mathbb{Z}}, \theta_{\mathbb{X}}, \theta_{\mathbb{Z}})$-*decomposable* if

$$\varphi\big(\theta_{\mathbb{X}}(x), y\big) = \sup_{z \in \mathbb{Z}} \Big\{ \varphi\big(\theta_{\mathbb{X}\times\mathbb{Z}}(x, z), y\big) \dot{+} \Big(- \varphi\big(\theta_{\mathbb{Z}}(z), y\big) \Big) \Big\} \; , \quad \forall (x, y) \in \mathbb{X} \times \mathbb{Y} \; . \tag{33}$$

Here is our result that provides nested formula for Fenchel-Moreau conjugates.

Proposition 5.6. *Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{Y}'$ be four sets and $g : \mathbb{Y} \to [-\infty, +\infty]$ be a numerical function. Let $\varphi : \mathbb{Y}' \times \mathbb{Y} \to [-\infty, +\infty]$ be $(\theta_{\mathbb{X}\times\mathbb{Z}}, \theta_{\mathbb{X}}, \theta_{\mathbb{Z}})$-decomposable as in Definition 5.5.*

Let us define the coupling $\Phi : \mathbb{X} \times (\mathbb{Y} \times \mathbb{Z}) \to [-\infty, +\infty]$ by

$$\Phi\big(x, (y, z)\big) = \varphi\big(\theta_{\mathbb{X}\times\mathbb{Z}}(x, z), y\big) \; , \quad \forall (x, y, z) \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \; , \tag{34}$$

and the function $G : \mathbb{Y} \times \mathbb{Z} \to [-\infty, +\infty]$ by

$$G(y, z) = g(y) \dotplus \varphi\big(\theta_{\mathbb{Z}}(z), y\big) \; , \quad \forall (y, z) \in \mathbb{Y} \times \mathbb{Z} \; . \tag{35}$$

Then, we have the following Nested Formula between Fenchel-Moreau conjugates:

$$G^\Phi = g^\varphi \circ \theta_{\mathbb{X}} \; . \tag{36}$$

Proof. We refer the reader to Appendix A for the details of the proof. □

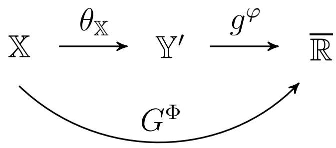

Figure 2: Representation of the Nested Formula (36)

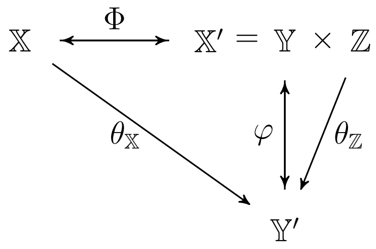

Figure 3: Representation of links between the mappings of Definition 5.5

6 Conclusion

Time Consistency is a notion discussed in economics (dynamic optimization, bargaining) and mathematics (dynamical risk measures, multi-stage stochastic optimization). We have gone through a selection of papers that are representative of the different fields; we have tried to separate the common core elements related to Time Consistency from the additional assumptions that make the specific contribution of each author. We have presented a framework of Weak Time Consistency which allows us to prove an equivalence with a Nested Formula, under minimal assumptions. By formulating the core skeleton axioms, we hope to have shed light on the notion of Time Consistency, often melted with other notions in the literature. We believe that this makes the notion more transparent and we showed that it opens the way for possible extensions. Indeed, we have established in Proposition 5.3 a nice relation between acceptance sets of the original mapping ρ and the "conditional" mapping F. In Proposition 5.6, we have put to light an intriguing relation that certainly needs further investigation.

Acknowledgements. The authors want to thank Université Paris-Est and Labex Bézout for the financial support. The first author particularly thanks them for the funding of his PhD program.

A Appendix

We provide here the proofs of two Propositions of Sect. 5.

A.1 Proof of Proposition 5.3

Proof. The proof goes in three steps as follows:

1. first, we show that $t \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Leftrightarrow F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}\ , \ \ \forall t \in \mathbb{T}$,

2. then we use the previous assertion to prove the two following statements:

$$\mathcal{A}_\rho \subset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Leftrightarrow \rho\big(F(t)\big) \leq \rho(t) \ , \quad \forall t \in \mathbb{T} \ , \tag{37a}$$

$$\mathcal{A}_\rho \supset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Leftrightarrow \rho\big(F(t)\big) \geq \rho(t) \ , \quad \forall t \in \mathbb{T} \ , \tag{37b}$$

3. finally, we bring all elements together to conclude.

We now detail each step.

1. We prove the implication $t \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Rightarrow F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ and the reverse statement $F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}} \Rightarrow t \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$ successively.

   - Let $t \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$ be given. By definition, t can be decomposed as $t = t_F \oplus t_\rho$ with $t_F \in \mathcal{A}_F$ and $t_\rho \in \mathcal{A}_{\rho_{|\mathbb{F}}}$. We successively obtain

     $$\begin{aligned} F(t) &= F(t_F) \oplus t_\rho \ , && \text{(as } t_\rho \in \mathbb{F} \text{ and } F \text{ is } (\mathbb{T},\mathbb{F})\text{-translation invariant)} \\ &\leq t_\rho \ , && \text{(as } t_F \in \mathcal{A}_F = \{t \in \mathbb{T} \mid F(t) \leq 0\}\text{)} \end{aligned}$$

     which leads to

     $$\begin{aligned} \rho\big(F(t)\big) &\leq \rho(t_\rho) \ , && \text{(by monotonicity of } \rho\text{)} \\ &\leq 0 \ , && \text{(by definition of } t_\rho \in \mathcal{A}_{\rho_{|\mathbb{F}}}\text{)} \end{aligned}$$

     and hence, $F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$.

   - We now assume that $F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ and recall that for all $t \in \mathbb{T}$, $F\big(t \ominus F(t)\big) = F(t) \ominus F(t) = 0$ by $(\mathbb{T}-\mathbb{F})$-translation invariance of the mapping F. The converse implication follows immediately from the decomposition $t = t \ominus F(t) \oplus F(t)$ since $F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ by assumption and $t \ominus F(t) \in \mathcal{A}_F$.

2. We prove statements (37a) and (37b) successively.

   - First, we focus on equation (37a):

     $$\mathcal{A}_\rho \subset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Leftrightarrow \rho\big(F(t)\big) \leq \rho(t) \ , \quad \forall t \in \mathbb{T} \ , \tag{38}$$

     We suppose that left hand side of this equation is satisfied, i.e. $\mathcal{A}_\rho \subset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$, and we show that it implies the right hand side of the equation. For that purpose, we fix $t \in \mathbb{T}$. We recall that $\rho(t) \in \mathbb{A} \subset \mathbb{F}$ by definition of the mapping $\rho : \mathbb{T} \to \mathbb{A}$ and assumption $(\mathbb{A}, \oplus) \subset (\mathbb{F}, \oplus)$ We have that $F(t) \ominus \rho(t) = F\big(t \ominus \rho(t)\big)$ by $(\mathbb{T}, \mathbb{F})$-translation invariance of the mapping$F$. As $t \ominus \rho(t) \in \mathcal{A}_\rho \subset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$ we get by item 1 just above that $F\big(t \ominus \rho(t)\big) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ and then $F(t) \ominus \rho(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$. This implies that

     $$\rho\big(F(t)\big) \ominus \rho(t) = \rho\big(F(t) \ominus \rho(t)\big) \leq 0 \ . \tag{39}$$

     Assume now that $\rho\big(F(t)\big) \leq \rho(t)$ for all $t \in \mathbb{T}$ and let $\tilde{t} \in \mathcal{A}_\rho$. Then by definition (23a) of an acceptance set, we got that $\rho(\tilde{t}) \leq 0$. It follows that $\rho\big(F(\tilde{t})\big) \leq 0$ and that $F(\tilde{t}) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ and so, by item 1 just above, that $\tilde{t} \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$.

- Second, we focus on Equation (37b)

$$\mathcal{A}_\rho \supset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \Leftrightarrow \rho\big(F(t)\big) \geq \rho(t) \ , \ \ \forall t \in \mathbb{T} \ , \tag{40}$$

We assume $\mathcal{A}_\rho \supset \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$. Let us fix $t \in \mathbb{T}$. Then, by adding and removing the term $F(t)$ we get

$$t \ominus \rho\big(F(t)\big) = \underbrace{t \ominus F(t)}_{\in \mathcal{A}_F} \oplus \underbrace{F(t) \ominus \rho\big(F(t)\big)}_{\in \mathcal{A}_{\rho|F}} \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}} \ . \tag{41}$$

It follows by left hand side of (37b) that $t \ominus \rho\big(F(t)\big)$ belongs to $\mathcal{A}_\rho$. That implies, taken together with the $(\mathbb{T}, \mathbb{A})$-translation invariance of the mapping $\rho : \mathbb{T} \to \mathbb{A}$

$$\rho(t) \ominus \rho\big(F(t)\big) = \rho\big(t \ominus \rho\big(F(t)\big)\big) \leq 0 \ . \tag{42}$$

To prove the reverse implication of Equation (37b), take $t \in \mathcal{A}_F \oplus \mathcal{A}_{\rho_{|\mathbb{F}}}$ and assume that $\rho\big(F(a)\big) \geq \rho(a)$. Using step 1, we have that $F(t) \in \mathcal{A}_{\rho_{|\mathbb{F}}}$ and we obtain that

$$\rho(t) \leq \rho\big(F(t)\big) \leq 0 \ , \tag{43}$$

which gives $t \in \mathcal{A}_\rho$ by definition (23a) of an acceptance set.

3. We finally bring all elements together. We know from Theorem 3.6 that the couple of mappings (A_ρ, F) is usual time consistent if and only if the subaggregator $S^{A_\rho,F}$ defined in (3) is a mapping increasing in its second argument and we have the nested formula $A_\rho(h,t) = S^{A_\rho,F}\big(h, F(t)\big)$.

   In this case, by Definition 3, we have that

$$S^{A_\rho,F}(h,f) = \big\{A_\rho(h,t) \mid F(t) = f\big\} \ , \ \ \forall (h,f) \in \mathbb{H} \times \mathbb{F} \ . \tag{44}$$

   As the set-valued mapping $S^{A_\rho,F}$ is a mapping, choosing one element $t \in \mathbb{T}$ such that $F(t) = f$ is sufficient to define the value of $S^{A_\rho,F}(h,f)$. We notice that, for each element $f \in \mathbb{F}$, the following statement holds true

$$F(t) = F(0) \oplus t \ . \tag{45}$$

   By $(\mathbb{T}, \mathbb{F})$-translation invariance property (22), we have that $F\big(f \ominus F(0)\big) = F(0) \oplus \big(f \ominus F(0)\big) = f$ for all $f \in \mathbb{F}$. We deduce that

$$S^{A_\rho,F}(h,f) = A_\rho\big(h, f \ominus F(0)\big) \ , \ \ \forall (h,f) \in \mathbb{H} \times \mathbb{F} \ . \tag{46}$$

   Hence, the nested formula $A_\rho(h,t) = S^{A_\rho,F}\big(h, F(t)\big)$ reads

$$\begin{aligned} A_\rho(h,t) &= A_\rho\big(h, F(t) \ominus F(0)\big) \ , && \text{(by (46))} \\ \rho(h \oplus t) &= \rho\big(h \oplus F(t) \ominus F(0)\big) \ , && \text{(by (25) that defines } A_\rho\text{)} \\ h \oplus \rho(t) &= h \oplus \rho\big(F(t) \ominus F(0)\big) \ , && \text{(by } (\mathbb{T}, \mathbb{A})\text{-translation invariance)} \\ \rho(t) &= \rho\big(F(t) \ominus F(0)\big) \ , && \text{(by compatibility of } \oplus \text{ with } \leq\text{)} \\ \rho(t) &= \rho\big(F(t)\big) && \text{(as } F(0) = 0.\text{)} \end{aligned}$$

   The fact that $\rho(t) = \rho\big(F(t)\big)$ taken together with both statements of Equations (37) gives the wanted result.

   This ends the proof.

□

A.2 Proof of Proposition 5.6

Proof. We have, for any $x \in \mathbb{X}$, the following equalities

$$G^{\Phi}(x) = \sup_{(y,z)\in\mathbb{Y}\times\mathbb{Z}} \Big\{ \Phi\big(x,(y,z)\big) \dotplus \big(- G(y,z) \big) \Big\} \ , \tag{47}$$

by Equation (31) that expresses the Φ-conjugate of G,

$$= \sup_{(y,z)\in\mathbb{Y}\times\mathbb{Z}} \Big\{ \varphi\big(\theta_{\mathbb{X}\times\mathbb{Z}}(x,z),y\big) \dotplus \big(- g(y) \big) \dotplus \Big(- \varphi\big(\theta_{\mathbb{Z}}(z),y\big) \Big) \Big\} \ , \tag{48}$$

by Equations (34) and (35) that express particular forms of Φ and G, and by the joint property (29a) of Moreau's additions,

$$= \sup_{y\in\mathbb{Y}} \Big\{ - g(y) \dotplus \sup_{z\in\mathbb{Z}} \Big\{ \varphi\big(\theta_{\mathbb{X}\times\mathbb{Z}}(x,z),y\big) \dotplus \big(- \varphi\big(\theta_{\mathbb{Z}}(z),y\big) \big) \Big\} \Big\} \ , \tag{49}$$

by property (29b) of Moreau's additions,

$$= \sup_{y\in\mathbb{Y}} \Big\{ - g(y) \dotplus \varphi\big(\theta_{\mathbb{X}}(x),y\big) \Big\} \ , \tag{50}$$

by Equation (33) that expresses the supremum,

$$= g^{\varphi}\big(\theta_{\mathbb{X}}(x)\big) \ , \tag{51}$$

by Definition 5.4 of a Fenchel-Moreau conjugate. This ends the proof. □

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