Sharp estimate of the mean exit time of a bounded domain in the zero white noise limit

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Abstract

We prove a sharp asymptotic formula for the mean exit time from an open bounded domain $D \subset \mathbb{R}^d$ for the overdamped Langevin dynamics

$$dX_t = -\nabla f(X_t)dt + \sqrt{2\varepsilon}\, dB_t$$

in the limit $\varepsilon \to 0$ and in the case when D contains a unique non degenerate minimum of f and $\partial_{\mathbf{n}} f > 0$ on ∂D. As a direct consequence, one obtains in the limit $\varepsilon \to 0$, a sharp asymptotic estimate of the smallest eigenvalue of the operator

$$L_\varepsilon = -\varepsilon\Delta + \nabla f \cdot \nabla$$

associated with Dirichlet boundary conditions on ∂D. The approach does not require $f|_{\partial D}$ to be a Morse function. The proof is based on results from [7,8] and a formula for the mean exit time from D introduced in the potential theoretic approach to metastability [4, 5].

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1 Setting and main results

Let us consider $(X_t)_{t\geq 0}$ the stochastic process solution to the overdamped Langevin dynamics in $\mathbb{R}^d$:

$$dX_t = -\nabla f(X_t)dt + \sqrt{2\varepsilon}\; dB_t, \tag{1}$$

where $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is the potential function, $\varepsilon > 0$ is the temperature and $(B_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion. The overdamped Langevin dynamics can be used for instance to describe the motion of the atoms of a molecule or the diffusion of impurities in a crystal (see for instance [22, Sections 2 and 3] or [6]). One of the major issues when trying to have access to the macroscopic evolution of the system from simulations made at the microscopic level is that the process (1) is metastable: it is trapped during long periods of time in some regions of the configuration space. This implies that it typically reaches a local equilibrium of these regions long before escaping from them. These regions are called metastable regions (see [3, Chapter 8]) and the move from one metastable region to another is typically associated with a macroscopic change of configuration of the system. The average time it takes for the process (1) to leave a metastable region is given by the Eyring-Kramers formula (see [14]). In this work, we would like to prove, in a typical geometric setting (see **[H-D]** below), that the average time it takes for the process (1) to leave a metastable region satisfies in the small temperature regime ($\varepsilon \to 0$) a kind of Eyring-Kramers formula even in the degenerate case when $\arg\min_{\partial D} f$ does not consists of a finite number of non degenerate critical points of $f|_{\partial D}$.

To this end, let us consider a C^∞ bounded open set $D \subset \mathbb{R}^d$ and introduce

$$\tau_{D^c} = \inf\{t \geq 0 | X_t \in D^c\} \tag{2}$$

where $D^c = \mathbb{R}^d \setminus D$, the first exit time from D. The framework we consider in this work is the following:

> Assumption **[H-D]**: $D \subset \mathbb{R}^d$ is a C^∞ bounded open set and $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$. The function f satisfies $\partial_{\mathbf{n}} f > 0$ on ∂D (where $\mathbf{n}$ is the unit outward normal to ∂D). Moreover, f has a unique critical point x_0 in D which is non degenerate and which satisfies $f(x_0) = \min_{\overline{D}} f$.

Under the assumption **[H-D]**, it is proved in [13, Theorem 4.1] that for any $x \in D$:

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x[\tau_{D^c}] = \min_{\partial D} f - f(x_0).$$

See also [21] for the study of the asymptotic behaviour of the law of $\varepsilon \log \tau_{D^c}$ in the limit $\varepsilon \to 0$ and [22] for formulas obtained with formal computations. Let us mention [1] for a review of the different techniques used to obtain asymptotic estimates on the mean exit time from a domain in the limit $\varepsilon \to 0$ in various geometric settings and for an extension of the Eyring-Kramers formulas in some degenerate cases when $D = \mathbb{R}^d$. In this paper, we prove a sharp asymptotic formula on the mean exit time from D in the limit $\varepsilon \to 0$. Our main result is the following.

Theorem 1. *Let us assume that the assumption* ***[H-D]*** *holds. Then, for any compact set $K \subset D$, it holds in the limit $\varepsilon \to 0$ and uniformly with respect to $x \in K$:*

$$\mathbb{E}_x[\tau_{D^c}] = \frac{(2\pi\varepsilon)^{\frac{d}{2}}}{\sqrt{\det \operatorname{Hess} f(x_0)} \displaystyle\int_{\partial D} \partial_{\mathbf{n}} f(\sigma) e^{-\frac{1}{\varepsilon} f(\sigma)} d\sigma} \, e^{-\frac{1}{\varepsilon} f(x_0)} \big(1 + O(\varepsilon)\big),$$

where $d\sigma$ is the Lebesgue measure on ∂D.

Remark 1. *Under some assumption on $f|_{\partial D}$, an asymptotic estimate of the term $\int_{\partial D} \partial_{\mathbf{n}} f(\sigma) e^{-\frac{1}{\varepsilon} f(\sigma)} d\sigma$ in the limit $\varepsilon \to 0$ can be obtained with Laplace's method.*

As a consequence of Theorem 1, one obtains an estimate in the limit $\varepsilon \to 0$ on the first eigenvalue of the infinitesimal generator of the diffusion (1)

$$L_\varepsilon = -\varepsilon \Delta + \nabla f \cdot \nabla. \tag{3}$$

with homogeneous Dirichlet boundary conditions on ∂D. Let us recall that since $D \subset \mathbb{R}^d$ is a C^∞ bounded open set and $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$, the operator L_ε with domain $H^2(D) \cap H^1_0(D)$ on $L^2(D, e^{-\frac{f(x)}{\varepsilon}} dx)$ is self-adjoint, positive and has compact resolvent, where $L^2(D, e^{-\frac{f(x)}{\varepsilon}} dx)$ is the completion of the space $C^\infty(\overline{D})$ for the norm

$$\phi \in C^\infty(\overline{D}) \mapsto \int_D |\phi|^2 e^{-\frac{1}{\varepsilon} f}.$$

Its smallest eigenvalue is denoted by $\lambda_\varepsilon > 0$. Theorem 1 together with [8, Corollary 1] (which is recalled in Section **2.2** below) imply the following estimates on λ_ε.

Corollary 1. *Let us assume that the assumption* ***[H-D]*** *holds. Then, in the limit $\varepsilon \to 0$:*

$$\lambda_\varepsilon = \frac{\sqrt{\det \operatorname{Hess} f(x_0)} \displaystyle\int_{\partial D} \partial_{\mathbf{n}} f(\sigma) e^{-\frac{1}{\varepsilon} f(\sigma)} d\sigma}{(2\pi\varepsilon)^{\frac{d}{2}}} \, e^{\frac{1}{\varepsilon} f(x_0)} \big(1 + O(\varepsilon)\big).$$

Let us mention that sharp estimates of the smallest eigenvalues of L_ε have been obtained in [9, 10, 16, 20] in the Dirichlet case and in [19] in the Neumann case when $f|_{\partial D}$ is a Morse function (i.e. when all the critical points of $f|_{\partial D}$ are non degenerate). When $D = \mathbb{R}^d$, we refer to [2, 4, 5, 15, 18, 23]. Corollary 1 gives a general formula on the asymptotic estimate of λ_ε which allows in particular, under the assumption **[H-D]**, to deal with the case when $f|_{\partial D}$ is not a Morse function. For example, direct consequences of Theorem 1 are the following:

- Let us assume that f is constant on ∂D: $f(z) \equiv f_1$ for all $z \in \partial D$. Then, for any compact set $K \subset D$, it holds:

$$\mathbb{E}_x[\tau_{D^c}] = \frac{(2\pi\varepsilon)^{\frac{d}{2}}}{\sqrt{\det \operatorname{Hess} f(x_0)} \int_{\partial D} \partial_{\mathbf{n}} f(\sigma) d\sigma} \, e^{\frac{1}{\varepsilon}(f_1 - f(x_0))}\big(1 + O(\varepsilon)\big),$$

in the limit $\varepsilon \to 0$ and uniformly with respect to $x \in K$. Moreover, one has in the limit $\varepsilon \to 0$

$$\lambda_\varepsilon = \frac{\sqrt{\det \operatorname{Hess} f(x_0)} \int_{\partial D} \partial_{\mathbf{n}} f(\sigma) d\sigma}{(2\pi\varepsilon)^{\frac{d}{2}}} \, e^{-\frac{1}{\varepsilon}(f_1 - f(x_0))}\big(1 + O(\varepsilon)\big).$$

- Let us assume that there exists $k \in \mathbb{N}^*$ such that $\arg\min_{\partial D} f = \{z_1, ..., z_k\}$ and for all $j \in \{1, ..., k\}$, z_j is a non degenerate critical point of $f|_{\partial D}$. Then, for any compact set $K \subset D$, it holds:

$$\mathbb{E}_x[\tau_{D^c}] = \sqrt{\pi\varepsilon} \sum_{j=1}^{k} \frac{\sqrt{\det \operatorname{Hess} f|_{\partial D}(z_j)}}{\partial_{\mathbf{n}} f(z_j) \sqrt{\det \operatorname{Hess} f(x_0)}} \, e^{\frac{1}{\varepsilon}(f(z_1) - f(x_0))}\big(1 + O(\varepsilon)\big)$$

in the limit $\varepsilon \to 0$ and uniformly with respect to $x \in K$. Moreover, one has in the limit $\varepsilon \to 0$

$$\lambda_\varepsilon = \frac{1}{\sqrt{\pi\varepsilon}} \sum_{j=1}^{k} \frac{\partial_{\mathbf{n}} f(z_j) \sqrt{\det \operatorname{Hess} f(x_0)}}{\sqrt{\det \operatorname{Hess} f|_{\partial D}(z_j)}} \, e^{-\frac{1}{\varepsilon}(f(z_1) - f(x_0))}\big(1 + O(\varepsilon)\big).$$

In particular, if $f|_{\partial D}$ is a Morse function, one recovers the results of [9,10,16] on the first eigenvalue λ_ε.

2 Change of coordinates in a neighborhood of ∂D

In this section, one constructs coordinates which will be useful for the computations in Section 4. The construction of these coordinates heavily depends on the assumption $\partial_{\mathbf{n}} f > 0$ on ∂D.

In all this section, we assume that the assumption **[H-D]** is satisfied.

2.1 Eikonal solution near ∂D

Let us start with the following lemma.

Lemma 1. *Let us assume that the assumption **[H-D]** holds. Then, there exists a neighborhood of ∂D in $\overline{D}$, denoted by $V_{\partial D}$, such that there exists $\Phi \in C^\infty(V_{\partial D}, \mathbb{R})$*

satisfying

$$\begin{cases} |\nabla \Phi|^2 = |\nabla f|^2 \text{ in } D \cap V_{\partial D} \\ \Phi = f \text{ on } \partial D \\ \partial_{\mathbf{n}} \Phi = -\partial_{\mathbf{n}} f \text{ on } \partial D. \end{cases} \tag{4}$$

Moreover, one has the following uniqueness results: if $\tilde{\Phi}$ is a C^∞ real valued function defined on a neighborhood $\tilde{V}$ of ∂D satisfying (4)*, then $\tilde{\Phi} = \Phi$ on $\tilde{V} \cap V_{\partial D}$. Finally, $V_{\partial D}$ can be chosen such that $\Phi > f$ on $V_{\partial D} \setminus \partial D$ and $\nabla(\Phi - f) \neq 0$ on $V_{\partial D}$.*

Proof. Let $z \in \partial D$. Using [11, Theorem 1.5] or [12, Section 3.2] and thanks to the fact that $\partial_{\mathbf{n}} f > 0$ on ∂D, there exists a neighborhood of z in $\overline{D}$, denoted by $\mathcal{V}_z$, such that there exists $\Phi \in C^\infty(\mathcal{V}_z, \mathbb{R})$ satisfying

$$\begin{cases} |\nabla \Phi|^2 = |\nabla f|^2 \text{ in } D \cap \mathcal{V}_z \\ \Phi = f \text{ on } \partial D \cap \mathcal{V}_z \\ \partial_{\mathbf{n}} \Phi = -\partial_{\mathbf{n}} f \text{ on } \partial D \cap \mathcal{V}_z. \end{cases}$$

Moreover, $\mathcal{V}_z$ can be chosen such that the following uniqueness result holds: if a function $\tilde{\Phi} \in C^\infty(\mathcal{V}_z, \mathbb{R})$ satisfies the previous equalities, then $\tilde{\Phi} = \Phi$ on $\mathcal{V}_z$. Now, one concludes using the fact that ∂D is compact and can thus it can be covered by a finite number of these neighborhoods $(\mathcal{V}_z)_{z \in \partial D}$. Finally, since $\partial_{\mathbf{n}}(\Phi - f) = -2\partial_{\mathbf{n}} f < 0$ on ∂D, $V_{\partial D}$ can be chosen such that $\Phi > f$ on $V_{\partial D} \setminus \partial D$ and $\nabla(\Phi - f) \neq 0$ on $V_{\partial D}$. ∎

2.2 Definition of the coordinate x_d

In this section, one defines coordinates near ∂D which will be convenient in the upcoming computations in Section **3**. Let us now consider Φ the solution to (4) on the neighborhood $V_{\partial D}$ of ∂D as introduced in Lemma 1. Let us define on $V_{\partial D}$:

$$f_+ = \frac{f + \Phi}{2} \text{ and } f_- = \frac{\Phi - f}{2}. \tag{5}$$

Using Lemma 1, it holds on $V_{\partial D} \setminus \partial D$: $f_- > 0$ and one has on $V_{\partial D}$:

$$\nabla f_- \cdot \nabla f_+ = 0. \tag{6}$$

Let us now consider $\delta > 0$ such that

$$V_\delta := \{x \in \overline{D},\, 0 \leq f_-(x) \leq \delta\} \subset V_{\partial D}.$$

For any $x \in V_\delta$, the dynamics

$$\begin{cases} \gamma_x'(t) = -\dfrac{\nabla f_-}{|\nabla f_-|^2}(\gamma_x(t)) \\ \gamma_x(0) = x \end{cases} \tag{7}$$

is well defined (since from Lemma 1, one has on $V_{\partial D}$, $\nabla f_- \neq 0$) and is such that $\gamma_x(t_x) \in \partial D$, where $t_x = \inf\{t,\ \gamma_x(t) \in \partial D\}$. This is indeed a consequence of the fact that $\frac{d}{dt} f_-(\gamma_x(t)) = -1 < 0$ on $[0, t_x)$.

Proposition 1. *The application*

$$\Theta : \begin{cases} V_\delta \to \partial D \times [0,\delta] \\ x \mapsto (\gamma_x(t_x), t_x) \end{cases}$$

defines a C^∞ diffeomorphism. The inverse application of Θ is

$$\Psi : (z, x_d) \in \partial D \times [0,\delta] \mapsto \gamma_z(-x_d).$$

Remark 2. *Let us mention that the application Ψ has been introduced locally in [16] and have also been used in [10].*

Let us now give some properties of the function Ψ which are used in the sequel. Using the fact that $\Psi(z, x_d) = \gamma_z(-x_d)$, one obtains that for all $z \in \partial D$ and $x_d \in [0,\delta]$:

$$\nabla_{x_d} \Psi(z, x_d) = \frac{d}{dx_d} \gamma_z(-x_d) = \frac{\nabla f_-(z, x_d)}{|\nabla f_-(z, x_d)|^2}. \tag{8}$$

Thus, one has for all $z \in \partial D$:

$$\nabla_{x_d} \Psi(z, 0) = -\frac{1}{\partial_{\mathbf{n}} f(z, 0)} \mathbf{n}, \tag{9}$$

where $\mathbf{n}$ is the unit outward normal to ∂D. Moreover, using the fact that $\Psi(z, 0) = (z, 0)$ for all $z \in \partial D$ and $\mathbf{n} = -\frac{\nabla x_d}{|\nabla x_d|}$ together with (9), it holds for all $u \in T_z \partial D$ and for all $v \in \mathbb{R}$:

$$d\Psi_{(z,0)}(u + v\mathbf{n}) = u + \frac{v}{\partial_{\mathbf{n}} f(z, 0)} \mathbf{n}, \tag{10}$$

and thus:

$$\operatorname{jac} \Psi(z, 0) = \frac{1}{\partial_{\mathbf{n}} f(z, 0)}, \tag{11}$$

where $\operatorname{jac} \Psi$ is the determinant of the jacobian matrix of Ψ. Finally, by construction (since $\frac{d}{dt} f_-(\gamma_x(t)) = -1$) $x_d(x) = f_-(x)$ and one has $\{x_d = 0\} = \partial D$, $\{x_d > 0\} = D \cap V_\delta$ and

$$V_\delta = \big\{x = \Psi(z, x_d) \in \overline{D},\ 0 \leq x_d \leq \delta\big\}. \tag{12}$$

2.3 Metric associated with the change of variable $x = \Psi(z, x_d)$

Let us consider $(\rho_k)_{k \in \{1,\dots,N\}} \in C^\infty(\partial D, [0,1])^N$ a partition of unity of ∂D:

$$\text{for all } y \in \partial D,\ \sum_{k=1}^{N} \rho_k(y) = 1 \tag{13}$$

such that for all $k \in \{1, ..., N\}$, there exist smooth coordinates $x' \in \mathbb{R}^{d-1}$ defined by a C^∞ mapping

$$\Gamma_k : \begin{cases} \operatorname{supp} \rho_k \to \mathbb{R}^{d-1} \\ \quad\quad z \mapsto x' \end{cases}. \tag{14}$$

The coordinates $x' \in \Gamma_k(\operatorname{supp} \rho_k)$ are then extended in a neighborhood of $\operatorname{supp} \rho_k$ in D, as constant along the integral curves of $\gamma'(t) = \frac{\nabla f_-}{|\nabla f_-|^2}(\gamma(t))$, for $t \in [0, \delta]$. The function $x \mapsto (x', x_d)$ (where, we recall, $x_d(x) = f_-(x)$) thus defines a smooth system of coordinates in a neighborhood V_k of $\operatorname{supp} \rho_k$ in $\overline{D}$. Let us define

$$\text{for all } (x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta], \quad \Upsilon_k(x', x_d) := \Psi\big(\Gamma_k^{-1}(x'), x_d\big) \tag{15}$$

where Ψ is introduced in Proposition 1. Notice that it holds for all $(x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta]$,

$$\operatorname{Jac} \Upsilon_k(x', x_d) := \operatorname{Jac} \Psi\big(\Gamma_k^{-1}(x'), x_d\big) \begin{pmatrix} \operatorname{Jac} \Gamma_k^{-1}(x') & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

where $\operatorname{Jac} \Upsilon_k$ is the jacobian matrix of Υ_k. In this system of coordinates, the metric tensor $G_k(x', x_d) = {}^t\operatorname{Jac} \Upsilon_k(x', x_d) \operatorname{Jac} \Upsilon_k(x', x_d)$ writes:

$$G_k : (x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta] \mapsto \begin{pmatrix} \tilde{G}_k(x', x_d) & 0 \\ 0 & (G_k)_{dd}(x', x_d) \end{pmatrix} \tag{17}$$

where $\tilde{G}_k$ is a C^∞ square matrix of size $d-1$ and $(G_k)_{dd}$ is a C^∞ positive function. Let us prove (17). Let us denote by $x' = (x'_1, ..., x'_{d-1})$. Since by construction, for all $(x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta]$, $f_-(\Upsilon_k(x', x_d)) = x_d$, one has:

$$\forall j \in \{1, ..., d-1\}, \ \nabla_{x'_j} \Upsilon_k(x', x_d) \cdot \nabla f_-(\Upsilon_k(x', x_d)) = 0. \tag{18}$$

Moreover, from (16) and (8), one has for all $(x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta]$,

$$\nabla_{x_d} \Upsilon_k(x', x_d) = \frac{\nabla f_-(\Upsilon_k(x', x_d))}{|\nabla f_-(\Upsilon_k(x', x_d))|^2}. \tag{19}$$

Then, from (18) and (19), it holds

$$\forall j \in \{1, ..., d-1\}, \ (G_k)_{j,d} = \nabla_{x'_j} \Upsilon_k \cdot \nabla_{x_d} \Upsilon_k = 0.$$

This proves (17). Furthermore, from (10) and (16), one has:

$$\text{for all } \ x' \in \Gamma_k(\operatorname{supp} \rho_k), \ (G_k)_{dd}(x', 0) = \frac{1}{\partial_{\mathbf{n}} f(x', 0)^2}. \tag{20}$$

Finally, a consequence of (6) is that $\frac{d}{dt} f_+(\gamma_x(t)) = 0$, where γ_x satisfies (7) and thus, in the system of coordinates (x', x_d), the functions f_+ and f write:

$$f_+(x', x_d) = f_+(x', 0) \text{ and } f(x', x_d) = f_+(x', 0) - x_d, \tag{21}$$

where with a slight abuse of notation, one denotes $f(\Upsilon_k(x', x_d))$ (resp. $f_+(\Upsilon_k(x', x_d))$) by $f(x', x_d)$ (resp. by $f_+(x', x_d)$).

3 Potential theory and mean exit time of D

3.1 Potential theory

Let us recall the main results from Potential theory which are used in this work. These results can be found for instance in [3]. Let us denote by $C = B(x_0, r_0) \subset D$ a closed ball centred at x_0 and of radius $r_0 > 0$ chosen such that $B(x_0, r_0) \cap V_\delta = \emptyset$ where V_δ is given by (12). Let h_{C,D^c} be the unique weak solution in $H^1(\mathbb{R}^d)$ of the elliptic boundary value problem

$$\begin{cases} L_\varepsilon v = 0 \text{ on } D \setminus C \\ \quad v = 0 \text{ on } D^c \\ \quad v = 1 \text{ on } C, \end{cases}$$

The function h_{C,D^c} is called the *equilibrium potential* of the *capacitor* (C, D^c) (as denoted in [4, Section 2]). From elliptic regularity estimates (see for instance [12, Theorem 5, Section 6.3]), the function h_{C,D^c} belongs to $C^\infty(\overline{D \setminus C})$. Therefore, it holds

$$h_{C,D^c} \in H^1(D) \cap C^\infty(\overline{D \setminus C}).$$

Using the Dynkin's formula (see for instance [17, Theorem 11.2]), one has for all $x \in \overline{D}$,

$$h_{C,D^c}(x) = \mathbb{P}_x[\tau_C < \tau_{D^c}], \tag{22}$$

where $\tau_C = \inf\{t \geq 0 | X_t \in C\}$ and τ_{D^c} is defined by (2). Let us denote by G_D be the Green function of L_ε associated with homogeneous Dirichlet boundary conditions on ∂D. The *equilibrium measure* e_{C,D^c} associated with (C, D^c) (see [4, Section 2] and more precisely the equation (2.10) there) is defined as the unique measure on ∂C such that

$$h_{C,D^c}(x) = \int_{\partial C} G_D(x, y) e_{C,D^c}(dy).$$

From [4, Section 2] (see equation (2.27) there), one has the following relation:

$$\int_{\partial C} \mathbb{E}_z[\tau_{D^c}]\, e^{-\frac{1}{\varepsilon} f(z)}\, e_{C,D^c}(dz) = \int_D e^{-\frac{1}{\varepsilon} f(x)}\, h_{C,D^c}(x)\, dx. \tag{23}$$

Let us now define, as in [4, Section 2] (see equation (2.13) there), the *capacity* associated with (C, D^c):

$$\mathrm{cap}_C(D^c) = \int_{\partial C} e^{-\frac{1}{\varepsilon} f(z)} e_{C,D^c}(dz). \tag{24}$$

3.2 A first asymptotic estimate on the mean exit time of D

The following results from [8, Corollary 1] and [7, Theorem 2] will be useful in the sequel.

Proposition 2. *Let us assume that the assumption **[H-D]** holds. Let $K \subset D$ be a compact set. Then, there exists $c > 0$ such that it holds in the limit $\varepsilon \to 0$ and uniformly with respect to $x \in K$:*

$$\lambda_\varepsilon \mathbb{E}_x[\tau_{D^c}] = 1 + O(e^{-\frac{c}{\varepsilon}}),$$

and

$$h_{C,D^c}(x) \geq 1 - e^{-\frac{c}{\varepsilon}},$$

where, we recall, for all $x \in \overline{D}$, $h_{C,D^c}(x) = \mathbb{P}_x[\tau_C < \tau_{D^c}]$ (see (22)*).*

Remark 3. *In [8, Corollary 1], the result on $\lambda_\varepsilon \mathbb{E}_x[\tau_{D^c}]$ is not stated with an error term. However, in view of the proof of [8, Corollary 1], the error term is $O(e^{-\frac{c}{\varepsilon}})$ and is uniform with respect to x in a compact subset of D.*

Proposition 2 implies that in the limit $\varepsilon \to 0$ and uniformly with respect to $x \in K$:

$$\mathbb{E}_x[\tau_{D^c}] = \mathbb{E}_{x_0}[\tau_{D^c}](1 + O(e^{-\frac{c}{\varepsilon}})). \tag{25}$$

We are now in position to obtain a first estimate on the mean exit time of D. Using (25), (23) and (24), for any compact set $K \subset D$, there exists $c > 0$ such that in the limit $\varepsilon \to 0$:

$$\mathbb{E}_{x_0}[\tau_{D^c}] = \frac{\displaystyle\int_D e^{-\frac{1}{\varepsilon}f(x)} h_{C,D^c}(x)dx}{\mathrm{cap}_C(D^c)}(1 + O(e^{-\frac{c}{\varepsilon}})).$$

Moreover, since $h_{C,D^c} \equiv 1$ on C, $h_{C,D^c} \leq 1$ on D, $f(x) \geq \max_{\overline{C}} f > f(x_0)$ for all $x \in D \setminus C$ and using Laplace's method (since x_0 is non degenerate), one obtains that there exits $c > 0$ such that in the limit $\varepsilon \to 0$:

$$\begin{aligned}\int_D e^{-\frac{1}{\varepsilon}f(x)} h_{C,D^c}(x)dx &= \int_C e^{-\frac{1}{\varepsilon}f(x)}\, dx + O(e^{-\frac{1}{\varepsilon}(f(x_0)+c)}) \\ &= \frac{(2\pi\varepsilon)^{\frac{d}{2}}}{\sqrt{\det \mathrm{Hess}\, f(x_0)}}\, e^{-\frac{1}{\varepsilon}f(x_0)}\big(1 + O(\varepsilon)\big).\end{aligned}$$

Thus, one has the following result.

Lemma 2. *Let us assume that the assumption **[H-D]** is satisfied. Then, in the limit $\varepsilon \to 0$:*

$$\mathbb{E}_{x_0}[\tau_{D^c}] = \frac{(2\pi\varepsilon)^{\frac{d}{2}}}{\sqrt{\det \mathrm{Hess}\, f(x_0)}\, \mathrm{cap}_C(D^c)}\big(1 + O(\varepsilon)\big), \tag{26}$$

where τ_{D^c} is defined by (2) *and* $\mathrm{cap}_C(D^c)$ *by* (24)*.*

To prove Theorem 1, it remains to give an estimate on $\mathrm{cap}_C(D^c)$ in the limit $\varepsilon \to 0$. This is the purpose of the next section.

4 Proofs of Theorem 1 and Corollary 1

In this section, one obtains sharp lower and upper bounds on the capacity $\text{cap}_C(D^c)$. From [4, Section 2], one has the following variational principle:

$$\begin{aligned}\text{cap}_C(D^c) &= \varepsilon \int_{D\setminus C} \big|\nabla h_{C,D^c}(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx \\ &= \inf_{h\in H_{C,D^c}} \varepsilon \int_{D\setminus C} \big|\nabla h(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx,\end{aligned} \tag{27}$$

where

$$H_{C,D^c} = \big\{h \in H^1(\mathbb{R}^d),\ h(x) = 1 \text{ for } x \in C,\ h(x) = 0 \text{ for } x \in D^c\big\}.$$

Formula (27) holds since the function h_{C,D^c} is a minimizer of the functional

$$h \in H_{C,D^c} \mapsto \varepsilon \int_{D\setminus C} \big|\nabla h(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx.$$

Using this variational principle, one can get a sharp upper bound on $\text{cap}_C(D^c)$ by choosing a suitable function $h \in H_{C,D^c}$.

4.1 Upper bound on $\text{cap}_C(D^c)$

In this section, one gets a sharp upper bound on $\text{cap}_C(D^c)$. Let V_δ be defined by (12) and let $h \in H_{C,D^c}$. From 27, one has

$$\text{cap}_C(D^c) \le \varepsilon \int_{V_\delta} \big|\nabla h(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx + \varepsilon \int_{D\setminus V_\delta} \big|\nabla h(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx. \tag{28}$$

From (13), (14), (15) and (17), one has:

$$\begin{aligned}&\varepsilon \int_{V_\delta} \big|\nabla h(x)\big|^2 e^{-\frac{1}{\varepsilon}f(x)}dx \\ &= \varepsilon \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\operatorname{supp}\rho_k)} \rho_k(\Gamma_k^{-1}(x')) \\ &\quad\times \int_0^\delta {}^t\tilde{\nabla} h(x',x_d) G_k(x',x_d)^{-1} \tilde{\nabla} h(x',x_d)\, e^{-\frac{1}{\varepsilon}f(x',x_d)} \operatorname{jac}\Upsilon_k(x',x_d)\, dx_d\, dx'\end{aligned} \tag{29}$$

where ${}^t\tilde{\nabla} = (\partial_{x'}, \partial_{x_d})$, Υ_k is defined by (15), G_k is the tensor metric associated with the change of variable $x = \Upsilon_k(x', x_d)$ (see (17)) and $\operatorname{jac}\Upsilon_k = \sqrt{\det G_k}$ is the jacobian of Υ_k.

Let us now consider the following function:

$$x_d \in [0,\delta] \mapsto g(x_d) = \frac{\displaystyle\int_0^{x_d} e^{-\frac{t}{\varepsilon}}dt}{\displaystyle\int_0^{\delta} e^{-\frac{t}{\varepsilon}}dt} = \frac{1-e^{-\frac{x_d}{\varepsilon}}}{1-e^{-\frac{\delta}{\varepsilon}}},$$

which satisfies $g(0)=0$ and $g(\delta)=1$. Let $h : V_\delta \to \mathbb{R}$ be such that

$$h \circ \Psi(z, x_d) := g(x_d), \text{ for all } (z, x_d) \in \partial D \times [0, \delta].$$

The function h is then extended by 1 in $D \setminus V_\delta$ and by 0 outside D. Thus, h belongs to H_{C,D^c} since $C \subset D \setminus V_\delta$. For all $k \in \{1, ..., N\}$ and for all $(x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta]$, denoting with a slight abuse of notation $h \circ \Upsilon_k$ by h, one has $h(x', x_d) = g(x_d)$ and then for any $(x', x_d) \in \Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta]$:

$$\partial_{x'} h(x', x_d) = 0 \text{ and } \partial_{x_d} h(x', x_d) = \frac{d}{dx_d} g(x_d).$$

From (17), (21), (28), and (29) together with the fact that $\nabla h = 0$ on $D \setminus V_\delta$, one has:

$$\begin{aligned} \operatorname{cap}_C(D^c) \leq \varepsilon \sum_{k=1}^{N} \int_{x' \in \Gamma_k(\operatorname{supp} \rho_k)} & \frac{\rho_k(\Gamma_k^{-1}(x'))\, e^{-\frac{1}{\varepsilon} f_+(x',0)}}{\varepsilon^2 \big(1 - e^{-\frac{\delta}{\varepsilon}}\big)^2} \\ & \times \int_0^\delta e^{-\frac{x_d}{\varepsilon}} (G_k)_{dd}(x', x_d)^{-1} \operatorname{jac} \Upsilon_k(x', x_d) dx_d\, dx'. \end{aligned}$$

Now let us notice that for any function $\varphi \in C^\infty(\Gamma_k(\operatorname{supp} \rho_k) \times [0, \delta], \mathbb{R}_+^*)$, one has in the limit $\varepsilon \to 0$:

$$\int_0^\delta \varphi(x', x_d) e^{-\frac{x_d}{\varepsilon}} dx_d = \varepsilon\, \varphi(x', 0)\big(1 + O(\varepsilon)\big), \tag{30}$$

uniformly with respect to $x' \in \Gamma_k(\operatorname{supp} \rho_k)$. Thus, applying (30) with $\varphi = (G_k)_{dd}^{-1} \operatorname{jac} \Upsilon_k$, it holds in the limit $\varepsilon \to 0$:

$$\begin{aligned} \operatorname{cap}_C(D^c) \leq \sum_{k=1}^{N} \int_{x' \in \Gamma_k(\operatorname{supp} \rho_k)} & \frac{\rho_k(\Gamma_k^{-1}(x')) e^{-\frac{1}{\varepsilon} f_+(x',0)}}{\big(1 - e^{-\frac{\delta}{\varepsilon}}\big)^2} \\ & \times (G_k)_{dd}(x', 0)^{-1} \operatorname{jac} \Upsilon_k(x', 0) dx' \,\big(1 + O(\varepsilon)\big). \end{aligned}$$

Finally, using (10), (16) and (20), it holds in the limit $\varepsilon \to 0$:

$$\operatorname{cap}_C(D^c) \leq \sum_{k=1}^{N} \int_{x' \in \Gamma_k(\operatorname{supp} \rho_k)} \rho_k(\Gamma_k^{-1}(x')) e^{-\frac{1}{\varepsilon} f(x',0)} \partial_{\mathbf{n}} f(x', 0) \operatorname{jac} \Gamma_k^{-1}(x') dx' \,\big(1 + O(\varepsilon)\big).$$

Therefore, since from (5) and Lemma 1, $f(x', 0) = f(x)$ for all $x = \Upsilon_k(x', 0) \in \partial D$, one has following result.

Lemma 3. *Let us assume that the assumption **[H-D]** is satisfied. Then, it holds in the limit $\varepsilon \to 0$:*

$$\operatorname{cap}_C(D^c) \leq \int_{\partial D} \partial_{\mathbf{n}} f(\sigma)\, e^{-\frac{1}{\varepsilon} f(\sigma)} d\sigma \,\big(1 + O(\varepsilon)\big), \tag{31}$$

where, we recall, $\operatorname{cap}_C(D^c)$ *is defined by* (24).

Let us now give a sharp lower bound on $\operatorname{cap}_C(D^c)$.

4.2 Lower bound on $\mathrm{cap}_C(D^c)$

In this section, one gets a sharp lower bound on $\mathrm{cap}_C(D^c)$. Let V_δ be defined by (12). Using (27), (13), (14), (15) and (17), one has:

$$\begin{aligned}\mathrm{cap}_C(D^c) &\geq \varepsilon \int_{V_\delta} \left|\nabla h_{C,D^c}(x)\right|^2 e^{-\frac{1}{\varepsilon}f(x)} dx \\ &\geq \varepsilon \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\mathrm{supp}\,\rho_k)} \rho_k(\Gamma_k^{-1}(x')) \int_0^\delta L_k(x',x_d)\, dx_d\, dx' \qquad (32)\end{aligned}$$

with

$$L_k(x',x_d) := \left|\partial_{x_d} h(x',x_d)\right|^2 (G_k)_{dd}(x',x_d)^{-1} e^{-\frac{1}{\varepsilon}f(x',x_d)}\, \mathrm{jac}\,\Upsilon_k(x',x_d).$$

Let us define for $k \in \{1,...,N\}$ and $(x',x_d) \in \Gamma_k(\mathrm{supp}\,\rho_k) \times [0,\delta]$:

$$\chi_k(x',x_d) := (G_k)_{dd}(x',x_d)^{-1}\, \mathrm{jac}\,\Upsilon_k(x',x_d). \qquad (33)$$

The function χ_k satisfies

$$\min_{\Gamma_k(\mathrm{supp}\,\rho_k)\times[0,\delta]} \chi_k > 0. \qquad (34)$$

Let us consider $k \in \{1,...,N\}$ and $x' \in \Gamma_k(\mathrm{supp}\,\rho_k)$. Then, it holds:

$$\begin{aligned}\int_0^\delta L_k(x',x_d)\, dx_d &= \int_0^\delta \left|\partial_t h_{C,D^c}(x',t)\right|^2 \chi_k(x',t)\, e^{\frac{t}{\varepsilon}}\, dt \\ &\geq \inf_{\substack{g\in H^1(0,\delta)\\ g(0)=0\\ g(\delta)=h_{C,D^c}(x',\delta)}} \int_0^\delta \left|\frac{d}{dt}g(t)\right|^2 \chi_k(x',t)\, e^{\frac{t}{\varepsilon}} dt. \qquad (35)\end{aligned}$$

Let us now prove that

$$\inf_{\substack{g\in H^1(0,\delta)\\ g(0)=0\\ g(\delta)=h_{C,D^c}(x',\delta)}} \int_0^\delta \left|\frac{d}{dt}g(t)\right|^2 \chi_k(x',t)\, e^{\frac{t}{\varepsilon}} dt = \int_0^\delta \left|\partial_t g^*_{x'}(t)\right|^2 \chi_k(x',t)\, e^{\frac{t}{\varepsilon}} dt, \qquad (36)$$

where

$$g^*_{x'}(t) = \frac{\displaystyle\int_0^t \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds}{\displaystyle\int_0^\delta \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds}\, h_{C,D^c}(x',\delta).$$

The set $K = \left\{g \in H^1(0,\delta), g(0) = 0 \text{ and } g(\delta) = h_{C,D^c}(x',\delta)\right\}$ is a closed convex subset of $H^1(0,\delta)$ and the functional

$$F : \theta \in H^1(0,\delta) \mapsto \int_0^\delta \left|\frac{d}{dt}\theta(t)\right|^2 \chi_k(x',t)\, e^{\frac{t}{\varepsilon}} dt$$

is continuous and from (34), it is strongly convex. Furthermore, since for all $u \in K$, $u(0) = 0$, there exists $C > 0$ such that for all $g \in K$,

$$\int_0^\delta g^2 \leq C \int_0^\delta \left|\frac{d}{dt} g(t)\right|^2.$$

Thus, using in addition (34), there exists $c > 0$ such that for all $g \in K$,

$$\int_0^\delta g^2 + \int_0^\delta \left|\frac{d}{dt} g(t)\right|^2 \leq c\, F(g). \tag{37}$$

Let us consider a sequence $(g_n)_{n\geq 0} \in K^{\mathbb{N}}$ such that $\lim_{n\to\infty} F(g_n) = \inf_K F$. Then, from (37), $(g_n)_{n\geq 0}$ is a bounded sequence in $H^1(0,\delta)$ and thus converges for the weak topology of $H^1(0,\delta)$ towards some $g \in H^1(0,\delta)$. Since F is continuous and convex on K, it is a lower semi-continuous function for the weak topology in $H^1(0,\delta)$. Therefore, $\inf_K F \leq F(g)$ and since $g \in K$, g is a minimizer of F on K. Finally, because F is strongly convex, g is the unique minimizer of F on K. Let $\alpha \in \mathbb{R}$ and $\varphi \in C_c^\infty(0,\delta)$. Then, it holds $g + \alpha\varphi \in K$ and thus

$$F(g) \leq F(g+\alpha\varphi) = F(g) + 2\alpha \int_0^\delta \frac{d}{dt} g(t)\, \frac{d}{dt}\varphi(t)\, \chi_k(x',t)\, e^{\frac{t}{\varepsilon}} dt + \alpha^2\, F(\varphi).$$

Thus, $g \in H^1(0,\delta)$ is a weak solution to the following one dimensional Dirichlet problem on $(0,\delta)$:

$$\begin{cases} \dfrac{d}{dt}\Big(e^{\frac{t}{\varepsilon}} \chi_k(x',t)\, \dfrac{d}{dt} g(t) \Big) = 0 \text{ on } (0,\delta), \\ g(0) = 0, \\ g(\delta) = h_{C,D^c}(x',\delta). \end{cases} \tag{38}$$

From (34), one can use the Lax-Milgram Theorem which implies that there exists a unique solution in $H^1(0,\delta)$ of (38). Clearly, this solution is given by

$$g^*_{x'}(t) = \frac{\displaystyle\int_0^t \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds}{\displaystyle\int_0^\delta \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds}\, h_{C,D^c}(x',\delta),$$

and thus $g = g^*_{x'}$. This concludes the proof of (36). Using (32), (35) and (36) together with the second statement in Proposition 2, there exists $c > 0$ such that in the limit

$\varepsilon \to 0$:

$$
\begin{aligned}
\mathrm{cap}_C(D^c) &\geq \varepsilon \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\mathrm{supp}\,\rho_k)} \rho_k(\Gamma_k^{-1}(x'))\, e^{-\frac{1}{\varepsilon}f_+(x',0)} \\
&\qquad \times \int_0^\delta \left|\partial_{x_d} g^*_{x'}(x_d)\right|^2 \chi_k(x',x_d) e^{\frac{1}{\varepsilon}x_d} dx_d\, dx' \\
&= \varepsilon \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\mathrm{supp}\,\rho_k)} \rho_k(\Gamma_k^{-1}(x'))\, h^2_{C,D^c}(x',\delta)\, e^{-\frac{1}{\varepsilon}f_+(x',0)} \\
&\qquad \times \int_0^\delta \frac{\chi_k(x',x_d)^{-1}}{\Big(\int_0^\delta \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds\Big)^2}\, e^{-\frac{1}{\varepsilon}x_d}\, dx_d\, dx' \\
&\geq \varepsilon (1-e^{-\frac{c}{\varepsilon}})^2 \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\mathrm{supp}\,\rho_k)} \rho_k(\Gamma_k^{-1}(x'))\, e^{-\frac{1}{\varepsilon}f_+(x',0)} \\
&\qquad \times \int_0^\delta \frac{\chi_k(x',x_d)^{-1}}{\Big(\int_0^\delta \chi_k(x',s)^{-1} e^{-\frac{s}{\varepsilon}} ds\Big)^2}\, e^{-\frac{1}{\varepsilon}x_d}\, dx_d\, dx'.
\end{aligned}
$$

Then, using (30), (33), (20), (10) and (16), one has in the limit $\varepsilon \to 0$:

$$
\mathrm{cap}_C(D^c) \geq \sum_{k=1}^{N} \int_{x'\in\Gamma_k(\mathrm{supp}\,\rho_k)} \rho_k(\Gamma_k^{-1}(x')) e^{-\frac{1}{\varepsilon}f(x',0)} \partial_{\mathbf{n}} f(x',0)\, \mathrm{jac}\, \Gamma_k^{-1}(x') dx' \left(1+O(\varepsilon)\right).
$$

Therefore, since from (5) and Lemma 1, $f(x',0) = f(x)$ for all $x = \Upsilon_k(x',0) \in \partial D$, one has the following lower bound on $\mathrm{cap}_C(D^c)$.

Lemma 4. *Let us assume that the assumption **[H-D]** is satisfied. Then, it holds in the limit $\varepsilon \to 0$:*

$$
\mathrm{cap}_C(D^c) \geq \int_{\partial D} \partial_{\mathbf{n}} f(\sigma)\, e^{-\frac{1}{\varepsilon}f(\sigma)} d\sigma \left(1+O(\varepsilon)\right). \tag{39}
$$

where, we recall, $\mathrm{cap}_C(D^c)$ *is defined by* (24).

Theorem 1 is then a consequence of (31) and (39) together with (26) and (25). Corollary 1 is a consequence of Theorem 1 and Proposition 2.

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