Complement to ‘Equilibrium large deviations for mean-field systems with translation invariance’

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ABSTRACT. We show that the metric space \((\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_P)\) introduced in the Appendix of [2] is separable and complete. Together with [2, Lemma A.3], this proves that the quotient topology on the orbit space \(\overline{\mathcal{P}}(\mathbb{R}^d)\) is Polish.

We recall that the space \(\mathcal{P}(\mathbb{R}^d)\) of Borel probability measures on \(\mathbb{R}^d\) endowed with the Prohorov metric \(d_P\) defined by Eq. (61) in [2] is separable and complete [1, Theorem 6.8, p. 73]. In the Appendix of [2], the quotient Prohorov metric \(\overline{d}_P\) is constructed on the orbit space \(\overline{\mathcal{P}}(\mathbb{R}^d)\), and it is proved that the associated metric topology coincides with the quotient topology.

Lemma 0.1 (Separability). The metric space \((\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_P)\) is separable.

Proof. This follows from the facts that \(\mathcal{P}(\mathbb{R}^d)\) is separable and that, by the definition of the quotient topology, the orbit map \(\rho: \mathcal{P}(\mathbb{R}^d) \to \overline{\mathcal{P}}(\mathbb{R}^d)\) is continuous. □

Lemma 0.2 (Completeness). The metric space \((\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_P)\) is complete.

Proof. Let \((\overline{\mu}_n)_{n \geq 1}\) be a Cauchy sequence in \(\overline{\mathcal{P}}(\mathbb{R}^d)\). There exists an increasing sequence of integers \((n_k)_{k \geq 1}\) such that, for any \(k \geq 1\), for all \(m \geq n_k\),

\[
\overline{d}_P(\overline{\mu}_{n_k}, \overline{\mu}_m) \leq 1/k^2.
\]

We now construct a sequence \((\nu_k)_{k \geq 1}\) in \(\mathcal{P}(\mathbb{R}^d)\) as follows:

- \(\nu_1 \in \mathcal{P}(\mathbb{R}^d)\) is such that \(\rho(\nu_1) = \overline{\mu}_{n_1}\);
- if \(\nu_k \in \mathcal{P}(\mathbb{R}^d)\) is such that \(\rho(\nu_k) = \overline{\mu}_{n_k}\), then combining Eq. (63) in [2] with (1), there exists \(\nu_{k+1} \in \mathcal{P}(\mathbb{R}^d)\) such that \(\rho(\nu_{k+1}) = \overline{\mu}_{n_{k+1}}\) and

\[
d_P(\nu_k, \nu_{k+1}) \leq \overline{d}_P(\overline{\mu}_{n_k}, \overline{\mu}_{n_{k+1}}) + 1/k^2 \leq 2/k^2.
\]

As a consequence, \((\nu_k)_{k \geq 1}\) is a Cauchy sequence in \(\mathcal{P}(\mathbb{R}^d)\) and therefore it converges to some \(\nu_* \in \mathcal{P}(\mathbb{R}^d)\). Using the continuity of \(\rho\), we deduce that \(\rho(\nu_k) = \overline{\mu}_{n_k}\) converges to \(\rho(\nu_*)\) in \(\overline{\mathcal{P}}(\mathbb{R}^d)\), so that the Cauchy sequence \((\overline{\mu}_n)_{n \geq 1}\) possesses a converging subsequence, and thus converges to \(\rho(\nu_*)\), which ends the proof. □

References


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