Complement to 'Equilibrium large deviations for mean-field systems with translation invariance'

Julien Reygner

Abstract. We show that the metric space $(\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_{\mathrm{P}})$ introduced in the Appendix of [2] is separable and complete. Together with [2, Lemma A.3], this proves that the quotient topology on the orbit space $\overline{\mathcal{P}}(\mathbb{R}^d)$ is Polish.

We recall that the space $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$ endowed with the Prohorov metric d_{P} defined by Eq. (61) in [2] is separable and complete [1, Theorem 6.8, p. 73]. In the Appendix of [2], the *quotient* Prohorov metric $\overline{d}_{\mathrm{P}}$ is constructed on the orbit space $\overline{\mathcal{P}}(\mathbb{R}^d)$, and it is proved that the associated metric topology coincides with the quotient topology.

Lemma 0.1 (Separability)**.** *The metric space* $(\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_{\mathrm{P}})$ *is separable.*

Proof. This follows from the facts that $\mathcal{P}(\mathbb{R}^d)$ is separable and that, by the definition of the quotient topology, the orbit map $\rho : \mathcal{P}(\mathbb{R}^d) \to \overline{\mathcal{P}}(\mathbb{R}^d)$ is continuous. □

Lemma 0.2 (Completeness)**.** *The metric space* $(\overline{\mathcal{P}}(\mathbb{R}^d), \overline{d}_{\mathrm{P}})$ *is complete.*

Proof. Let $(\overline{\mu}_n)_{n\geq 1}$ be a Cauchy sequence in $\overline{\mathcal{P}}(\mathbb{R}^d)$. There exists an increasing sequence of integers $(n_k)_{k\geq 1}$ such that, for any $k \geq 1$, for all $m \geq n_k$,

$$\overline{d}_{\mathrm{P}}(\overline{\mu}_{n_k}, \overline{\mu}_m) \leq 1/k^2. \tag{1}$$

We now construct a sequence $(\nu_k)_{k\geq 1}$ in $\mathcal{P}(\mathbb{R}^d)$ as follows:

- $\nu_1 \in \mathcal{P}(\mathbb{R}^d)$ is such that $\rho(\nu_1) = \overline{\mu}_{n_1}$;
- if $\nu_k \in \mathcal{P}(\mathbb{R}^d)$ is such that $\rho(\nu_k) = \overline{\mu}_{n_k}$, then combining Eq. (63) in [2] with (1), there exists $\nu_{k+1} \in \mathcal{P}(\mathbb{R}^d)$ such that $\rho(\nu_{k+1}) = \overline{\mu}_{n_{k+1}}$ and $d_{\mathrm{P}}(\nu_k, \nu_{k+1}) \leq \overline{d}_{\mathrm{P}}(\overline{\mu}_{n_k}, \overline{\mu}_{n_{k+1}}) + 1/k^2 \leq 2/k^2$.

As a consequence, $(\nu_k)_{k\geq 1}$ is a Cauchy sequence in $\mathcal{P}(\mathbb{R}^d)$ and therefore it converges to some $\nu_* \in \mathcal{P}(\mathbb{R}^d)$. Using the continuity of ρ, we deduce that $\rho(\nu_k) = \overline{\mu}_{n_k}$ converges to $\rho(\nu_*)$ in $\overline{\mathcal{P}}(\mathbb{R}^d)$, so that the Cauchy sequence $(\overline{\mu}_n)_{n\geq 1}$ possesses a converging subsequence, and thus converges to $\rho(\nu_*)$, which ends the proof. □

References

[1] P. Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York, second edition, 1999.

[2] J. Reygner. Equilibrium large deviations for mean-field systems with translation invariance. *Ann. Appl. Probab.*, 28(5):2922–2965, 2018.

E-mail address: julien.reygner@enpc.fr