On Melan's Theorem in Temperature-Dependent Viscoplasticity

Michaël Peigney

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Abstract In plasticity, Melan's theorem is a well-known result that is both of theoretical and practical importance. That theorem applies to elastic-plastic structures under time-dependent loading histories, and gives a sufficient condition for the plastic dissipation to remain bounded in time. That situation is classically referred to as shakedown. Regarding fatigue, shakedown corresponds to the most favorable case of high-cycle fatigue. The original Melan's theorem rests on the assumption that the material properties remain constant in time, independently on the applied loading. Extending Melan's theorem to time fluctuating elastic moduli is a long standing issue. The main motivation is to extend the range of applications of Melan's theorem to thermomechanical loading histories with large temperature fluctuations: In such case, the variation of the elastic properties with the temperature cannot be neglected. In this contribution, an extension of Melan's theorem to elastic-viscoplastic materials with time-periodic elastic moduli is presented. Such a time-dependence may for instance result from time-periodic temperature variations. An illustrative example is presented and supported by numerical results obtained from incremental analysis.

1 Introduction

For elastic-perfectly plastic structures under prescribed loading histories, the well-known Melan's theorem [1, 2, 3] gives a sufficient condition for the evolution to become elastic in the large-time limit. That situation is classically referred to as *shakedown*. Intuitively, shakedown means that the plastic strain tends to a limit as time tends to infinity. The Melan's theorem has the distinctive property of being path-independent, i.e. independent on the initial state of the structure. For a parameterized loading history, Melan's theorem gives bounds on the domain of load pa-

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rameters for which shakedown occurs. Regarding fatigue design, shakedown corresponds to the most beneficial regime of high-cycle fatigue, as opposed to the regime of low-cycle fatigue which typically occurs if the plastic strain does not converge towards a stabilized value [4]. The shakedown theory has been the object of numerous developments, regarding both extensions of the original theorem to various nonlinear behaviors [5, 6, 7, 8, 9, 10] and numerical methods for assessing the shakedown limits in the space of load parameters [11, 12, 13, 14, 15, 16, 17].

This chapter is concerned with extensions of the Melan's theorem to situations in which the elastic moduli are fluctuating in time, for instance as a result of imposed temperature variations. Whereas the case of temperature-dependent yield limits is well understood [18], the case of temperature-dependent elastic moduli remains a long standing issue and has been the object of several conjectures [19, 20, 21, 22]. The main difficulty is that the proof used in the original Melan's theorem – as well as in most of its knows extensions – crucially relies on some monotonicity properties that are lost when the elastic moduli are allowed to vary in time. For instance, in the case of constant elastic moduli, the distance between two solutions (as measured by the energy norm) is always decreasing with time [6], which is no longer true when the elastic moduli vary in time (see. [23] for some example). A shakedown theorem has recently been proposed for elastic-perfectly plastic materials with time-periodic elastic properties [23]. The statement and proof of that theorem differ significantly from the case of constant material properties. A salient result is that time fluctuations of the elastic moduli need to be not too large for shakedown to occur in a path-independent fashion.

In this contribution, we aim at extending the result of [23] to elastic-viscoplastic materials with time fluctuating elastic moduli. The proof presented in [23] for elastic-perfectly plastic materials makes use of the fact that the stress remains in the elasticity domain, which is not necessarily the case in viscoplasticity. In particular, the initial residual stress can be chosen as arbitrarily large, so that the stress is expected to remain outside (and possibly far away from) the elasticity domain – at least on some time interval. This chapter is organized as follows: In Sect. 2, starting from the local constitutive relations and the equilibrium equations, we derive the differential equation that governs quasistatic evolutions of the residual stress. We comment on the special (and important) case of elastic solutions. Sect. 3 is devoted to the statement and proof of a shakedown theorem for elastic-viscoplastic materials with time-periodic material properties, that is the main result of this chapter. An illustrative example is presented in Sect. 4 and supported by numerical results provided by incremental analysis. Some concluding remarks follow.

2 Quasistatic evolutions of an elastic-viscoplastic medium

Consider an elastic-viscoplastic body occupying a domain Ω in the reference configuration. Under the assumption of infinitesimal strains, the strain tensor ε is derived from the displacement u by

$$\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u).$$

The total strain ε, stress σ and plastic strain ε^p at position x and time t satisfy the constitutive equations

$$\varepsilon(x,t) = L(x,t) : \sigma(x,t) + \varepsilon^\theta(x,t) + \varepsilon^p(x,t), \tag{1}$$

$$\dot{\varepsilon}^p(x,t) = \phi'(\sigma(x,t),x,t), \tag{2}$$

where ϕ is the dissipation potential, taken in the form

$$\phi(\sigma,x,t) = \frac{\alpha}{2}|\sigma - P_{\mathscr{C}(x,t)}\sigma|^2. \tag{3}$$

In (1), L is the (symmetric positive definite) elastic moduli tensor and ε^θ is the thermal strain tensor. The double product : in (1) denotes contraction with respect to the last two indexes, i.e. $(L:\sigma)_{ij} = \sum_{k,l} L_{ijkl}\sigma_{lk}$.

In (3), $\mathscr{C}(x,t)$ is the elasticity domain of the material (assumed to be closed and convex), $\alpha > 0$ is a viscosity parameter (assumed to be independent on (x,t) for simplicity) and $P_{\mathscr{C}(x,t)}$ denotes the projection of $\mathscr{C}(x,t)$. The norm $|\cdot|$ in (3) is defined by $|\sigma| = \sqrt{\sum_{i,j}\sigma_{ij}^2}$ for any symmetric tensor σ.

As mentioned in Sect. 1, the space and time dependence of L, ε^θ and $\mathscr{C}$ may reflect imposed variations of the temperature. For instance, the elastic moduli L of most materials depend on the temperature θ, what can be written as $L = L(\theta)$. For imposed variations $\theta(x,t)$ of the temperature, the elastic moduli tensor vary as $L(\theta(x,t))$ and can thus be regarded as a function of space and time.

Assuming quasi-static evolutions, the stress field σ satisfies the equilibrium equations

$$\mathrm{div}\,\sigma + f = 0 \text{ in } \Omega,\ \sigma\cdot n = T \text{ on } \partial\Omega_T, \tag{4}$$

where $f(x,t)$ are body forces imposed in the domain Ω and $T(x,t)$ are tractions prescribed on a part $\partial\Omega_T$ of the boundary $\partial\Omega$. Prescribed displacements $v(x,t)$ are imposed on $\partial\Omega_u = \partial\Omega - \partial\Omega_T$.

2.1 Evolution equation for the residual stress

We now use Eqs. (1-4) to derive the equation governing the evolution of the stress field. The space $\mathcal{E}$ of stress fields is chosen as a subspace of symmetric second-order tensor fields with square-integrable components, which is known to be a Hilbert space for the scalar product

$$\langle \sigma, \sigma' \rangle = \int_\Omega \sigma(x) : \sigma'(x) d\omega.$$

The associated norm is denoted by $\| \ \|$, i.e. $\|\sigma\| = \sqrt{\langle \sigma, \sigma \rangle}$.

Consider the so-called *fictitious elastic response* (u^E, σ^E), i.e. the response of the system if it were purely elastic, defined by

$$\begin{aligned}
\varepsilon^E &= L : \sigma^E + \varepsilon^\theta, \\
\varepsilon^E &= \frac{1}{2}(\nabla u^E + \nabla^T u^E), \\
\operatorname{div} \sigma^E + f &= 0 \text{ in } \Omega, \\
\sigma^E \cdot n &= T \text{ on } \partial\Omega_T, \\
u^E &= v \text{ on } \partial\Omega_u.
\end{aligned} \tag{5}$$

The stress field σ can be written as $\sigma = \sigma^E + \rho$ where ρ is the residual stress field and belongs to the vectorial space $\mathcal{H} \subset \mathcal{E}$ of self-equilibrated fields, defined by

$$\mathcal{H} = \{\rho \in \mathcal{E} : \operatorname{div} \rho = 0 \text{ in } \Omega,\ \rho \cdot n = 0 \text{ on } \partial\Omega_T\}. \tag{6}$$

Let $\mathcal{K}_0(t)$ and $\mathcal{K}(t)$ be the convex subsets of $\mathcal{E}$ defined as

$$\mathcal{K}_0(t) = \{\sigma \in \mathcal{E} : \sigma(x,t) \in \mathscr{C}(x,t)\ \forall x \in \Omega\},\ \mathcal{K}(t) = \mathcal{K}_0(t) - \sigma^E(t). \tag{7}$$

The set $\mathcal{K}_0(t)$ is the set of stress fields that are everywhere in the elasticity domain of the material. The set $\mathcal{K}(t)$ is the translated of $\mathcal{K}_0$ by $-\sigma^E(t)$. Note that $\mathcal{K}_0$ is independent on time t if the yield parameters are. Under suitable regularity assumptions on (f, T, v), it can proved that the sets $\mathcal{H}$ and $\mathcal{K}(t)$ are closed in $\mathcal{E}$ [24].

For an arbitrary $\rho' \in \mathcal{K}(t)$, it follows from (1) that

$$\int_\Omega (\rho' - \rho) : \frac{d(\varepsilon - \varepsilon^E)}{dt}\, d\omega = \int_\Omega (\rho' - \rho) : \frac{d}{dt}(L : \rho)\, d\omega + \int_\Omega (\rho' - \rho) : \phi'(\sigma)\, d\omega. \tag{8}$$

Using (5-6) together with the principle of virtual power shows that the left-hand side of (8) is equal to zero. Hence

$$-\int_\Omega (\rho' - \rho) : \frac{d}{dt}(L(x,t) : \rho)\, d\omega = \int_\Omega (\rho' - \rho) : \phi'(\sigma, x, t)\, d\omega. \tag{9}$$

The function

$$\Phi(\sigma, t) = \int_\Omega \phi(\sigma, x, t)\, d\omega$$

is convex, positive, and vanishes on $\mathcal{K}_0(t)$. It is a classical result [25] that

$$\Phi(\sigma, t) = \frac{\alpha}{2}\|\sigma - P_{\mathcal{K}_0(t)}\sigma\|^2,\ \Phi'(\sigma, t) = \alpha(\sigma - P_{\mathcal{K}_0(t)}\sigma) \tag{10}$$

where $P_{\mathcal{K}_0(t)} : \mathcal{E} \mapsto \mathcal{E}$ denotes the projection on $\mathcal{K}_0(t)$ (for the scalar product $\langle \cdot, \cdot \rangle$). It follows from (10) that

$$\int_{\Omega} (\rho' - \rho) : \phi'(\sigma)\, d\omega = \langle \rho' - \rho, \Phi'(\sigma) \rangle = \langle \rho' - \rho, \alpha(\sigma - P_{\mathcal{K}_0(t)}\sigma) \rangle.$$

Further observing that $\sigma - P_{\mathcal{K}_0(t)}\sigma = \rho - P_{\mathcal{K}(t)}\rho$, we obtain from (9) that ρ satisfies

$$-\frac{d}{dt}(L(x,t)\rho) \in \alpha(\rho - P_{\mathcal{K}(t)}\rho) + \mathcal{H}^{\perp} \tag{11}$$

where $\mathcal{H}^{\perp}$ is the orthogonal of $\mathcal{H}$ in $\mathcal{E}$.

Eq. (11) can be simplified by projecting it on $\mathcal{H}$. To that purpose, set $\mathcal{L}(t) = \pi L$ where $\pi : \mathcal{E} \mapsto \mathcal{H}$ is the orthogonal projector on $\mathcal{H}$. Eq. (11) becomes

$$-\frac{d}{dt}(\mathcal{L}(t)\rho) = \alpha(\rho - \pi P_{\mathcal{K}(t)}\rho). \tag{12}$$

Since the elastic moduli tensor $L(x,t)$ is symmetric positive definite, it can easily be verified that $\mathcal{L}(t)$ is self-adjoint and positive definite. Starting from a given initial state $\rho(t=0)$, the evolution of the stress field in $\mathcal{H}$ is governed by the ordinary differential equation (12). The uniqueness of the stress rate $\dot{\rho}$ has been proved in [26].

2.2 *Elastic solutions*

In the following we study the asymptotic behavior of solutions to (12) as $t \to \infty$. We only consider the case where $\mathcal{L}(t)$, $\mathcal{K}(t)$ are periodic in time (with the same period T) and the dimension of $\mathcal{H}$ is finite. A central role is played by elastic solutions of (12), i.e. solutions without any plastic yielding. Such an elastic solution $\rho^*(t)$ necessarily lies in $\mathcal{K}(t)$ at each time t, and therefore satisfies

$$\frac{d}{dt}(\mathcal{L}(t)\rho^*(t)) = 0,\ \rho^*(t) \in \mathcal{K}(t) \cap \mathcal{H}\ \forall t \in [0,T] \tag{13}$$

Our main objective is to examine conditions under which *every* solution of (12) converges towards an elastic solution in the large time limit. A first requisite is obviously that elastic solutions do exist, i.e. that

$$\bigcap_{0 \le t \le T} \mathcal{L}(t)\mathcal{K}(t) \neq \emptyset. \tag{14}$$

Even when the elastic moduli are time-independent, the condition (14) is not sufficient to obtain results on the asymptotic behavior. Loosely speaking, a minimal requirement is the existence of an elastic solution that remains 'strictly inside' the elastic domain. In the case of time-independent elastic moduli, that notion of 'strictly inside' is captured by the Melan's condition defined as follows

Melan's condition (standard form): There exists an elastic solution ρ^* and some $m > 1$ such that $\rho^*(x,t) + m\sigma^E(x,t) \in \mathscr{C}(x,t)\ \forall (x,t) \in \Omega \times [0,T]$.

Let $B(\rho^*, r)$ denotes the ball of center ρ^* and radius r in $\mathcal{E}$ (for the norm $\|\cdot\|$). In the following, we consider a strong version of Melan's condition, defined as follows:

Melan's condition (strong version): There exists an elastic solution ρ^* and some $r > 0$ such that $\mathcal{H} \cap B(\rho^*(t), r) \in \mathcal{K}(t)\ \forall t \in \times [0,T]$.

When the dimension of $\mathcal{H}$ is finite, the strong and standard versions of Melan's condition are equivalent. This is no longer true in infinite dimension (the strong version, as its names suggests, is more restrictive).

3 Shakedown theorem

Let us fix some notations and assumptions to be used in the remainder:

- $\mu(t)$ denotes an arbitrary non-negative differentiable function such that $\inf_t \mu(t) > 0$. We set
$$\mathcal{L}_0(t) = \mathcal{L}(t)/\mu(t),\ \mathcal{M}_0(t) = \mathcal{L}_0^{-1}(t) = \mu(t)\mathcal{M}(t). \tag{15}$$
with $\mathcal{M}(t) = \mathcal{L}(t)^{-1}$.
- The set $\pi\mathcal{K}(t)$ remains bounded[1] i.e. there exists a constant M such that
$$\|\pi\rho'\| \leq M \text{ for all } t \text{ and } \rho' \in \mathcal{K}(t). \tag{16}$$
- The elastic operator $\mathcal{L}(t)$ is assumed to remain bounded in $\mathcal{H}$, i.e. there exists a constant C such that
$$|||\mathcal{L}(t)||| \leq C \tag{17}$$
for all $t \in [0,T]$. In (17), $|||\cdot|||$ denotes the norm operator in $\mathcal{H}$, i.e. $|||\mathcal{L}(t)||| = \sup_{\rho' \in \mathcal{H}, \|\rho'\|=1} \|\mathcal{L}(t)\rho'\|$. We note that (17) is satisfied if the local elastic moduli $L(x,t)$ remain bounded [23]. Similarly, the operator $\mathcal{M}(t)$ is assumed to be bounded.
- We introduce a measure $\gamma(0,T)$ of the time-fluctuations of the elastic moduli, defined as
$$\gamma(0,T) = \int_0^T |||\dot{\mathcal{M}}_0(t)|||\, dt = \int_0^T |||\mu(t)\dot{\mathcal{M}}(t) + \dot{\mu}(t)\mathcal{M}(t)|||\, dt. \tag{18}$$

The objective of this section is to prove Theorem 1 below.

[1] It can observed that $\pi\mathcal{K}(t)$ is bounded if the $\mathscr{C}(x,t)$ is.

Theorem 1. *If*

(i) the Melan's condition is satisfied by some (ρ^*, r)*;*

(ii) the elastic moduli are such that

$$\frac{\gamma(0,T)}{\inf \mu} < \frac{r}{2CM};$$

then the residual stress converges towards an elastic solution, whatever the initial state is.

In that theorem, condition (ii) sets restriction on the time variations of the elastic moduli. Setting such a restriction is necessary to obtain global convergence results: One can indeed find counterexamples in which condition (i) is fulfilled and some solutions of (12) do not become elastic in the large time limit [23].

The statement of Theorem 1 above is quite similar to that obtained in perfect plasticity [23]. The proof, however, is more complicated and detailed in the following. To clarify the exposition, the proof is broken down in 3 separate steps, covered by Sect. 3.1-3.3 below. Compared to perfect plasticity, an additional difficulty is that the stress is not restricted to remain in the elasticity domain. For instance, the initial stress can be chosen as arbitrarily large. It can be proved, however, that the stress is bounded at large time: This is the object of Lemma 1, Sect. 3.1. The next step consists in proving that, for large time, the variation of some elastic energy is controlled by the plastic dissipation, in a sense that is defined in Lemma 2, Sect. 3.2. The claimed result follows from those two lemmas, as detailed in Sect. 3.3.

In all that follows, $\rho(t)$ denotes an arbitrary solution of (12).

3.1 Bound on the stress field

Let $\eta(t) = \mathcal{L}(t)\rho(t)$. The object of the following lemma is to prove that, for large time, $\eta(t)$ is bounded by some constant M' that can be chosen as arbitrarily close to CM.

Lemma 1. *For any* $M' > CM$*, there exists* $t_0 \geq 0$ *such that* $\|\eta(t)\| \leq M'$ *for all* $t \geq t_0$.

Proof. We have

$$\begin{aligned} \frac{d}{dt}\|\eta(t)\|^2 &= -2\alpha\langle \rho - \pi P_{\mathcal{K}(t)}\rho, \eta(t)\rangle \\ &= -2\alpha\langle \eta(t), \mathcal{M}(t)\eta(t)\rangle + 2\alpha\langle \pi P_{\mathcal{K}(t)}\rho, \eta(t)\rangle. \end{aligned} \tag{19}$$

Since $\mathcal{L}(t)$ is symmetric positive definite, we have $|||\mathcal{L}(t)||| = \max_i \lambda_i$ where $\{\lambda_i\}$ are the eigenvalues of $\mathcal{L}(t)$. The relation (17) implies that $\lambda_i \leq C$ for all i. Since $\mathcal{M}(t)$ is positive definite with eigenvalues $\{1/\lambda_i\}$, it follows that

$$\langle\eta(t),\mathcal{M}(t)\eta(t)\rangle \geq \min_i \frac{1}{\lambda_i}\|\eta(t)\|^2 \geq \frac{\|\eta(t)\|^2}{C}.$$

Moreover, using Cauchy-Schwarz inequality together with (16), we find

$$2\alpha\langle\pi P_{\mathcal{K}(t)}\rho,\eta(t)\rangle \leq 2\alpha\|\pi P_{\mathcal{K}(t)}\rho\|\cdot\|\eta(t)\| \leq 2\alpha M\|\eta(t)\|.$$

Substituting in (19) gives

$$\frac{d}{dt}\|\eta(t)\|^2 \leq -2\frac{\alpha}{C}\|\eta(t)\|^2 + 2\alpha M\|\eta(t)\|. \tag{20}$$

Setting $G(t) = \max(M^2C^2, \|\eta(t)\|^2)$, Eq. (20) implies that G is decreasing with time t. It can indeed easily be verified from (20) that G is right-differentiable and satisfies $G'_+(t) \leq 0$ for all t, where $G'_+(t)$ is the right-derivative of G.

If $\|\eta(t_0)\| \leq CM$ for some time t_0, then it directly follows from the monotonicity of G that $\|\eta(t)\| \leq CM$ for any $t \geq t_0$, which proves the claim. Now consider the case where $\|\eta(t)\| > CM$ for all time t. Dividing (20) by $\|\eta(t)\|$, we have

$$\frac{1}{\|\eta(t)\|}\frac{d}{dt}\|\eta(t)\|^2 = 2\frac{d}{dt}\|\eta(t)\| \leq -2\frac{\alpha}{C}\|\eta(t)\| + 2\alpha M. \tag{21}$$

Using the differential form of Gronwall's lemma, (21) implies that

$$\|\eta(t)\| \leq \|\eta(t_0)\|e^{-\alpha t/C} + CM(1 - e^{-\alpha t/C}). \tag{22}$$

The right-hand side of (22) varies between $\|\eta(t_0)\|$ and CM in a monotonic fashion. Choosing for instance

$$t_0 = \frac{C}{\alpha}|\log\frac{\|\eta(0)\| - CM}{M' - CM}|,$$

Eq. (22) implies that $\|\eta(t)\| \leq M'$ for all $t \geq t_0$, which proves the claim. $\square$

3.2 Variation of the elastic energy

Consider the positive function f defined as

$$f(t) = \frac{1}{2}\langle\tau(t),\mathcal{L}_0(t)\tau(t)\rangle. \tag{23}$$

where $\tau(t) = \mu(t)(\rho(t) - \rho^*(t))$. The function f is referred to as the elastic energy. The aim of this section is to establish Lemma 2 below, which bounds the variation $f(a+T) - f(a)$ for large time a.

Lemma 2. *Let (M', t_0) satisfying Lemma 1. For any $a \geq t_0$ we have*

$$f(a+T)-f(a) \leq \left(2M'\gamma(0,T)-r\inf\mu\right)\int_a^{a+T}\|\dot{\eta}(t)\|dt. \tag{24}$$

Proof. We have

$$\dot{f}(t)=\langle\tau(t),\mathcal{L}_0(t)\dot{\tau}(t)\rangle+\frac{1}{2}\langle\tau(t),\dot{\mathcal{L}}_0(t)\tau(t)\rangle. \tag{25}$$

Recalling that $d(\mathcal{L}(t)\rho^*(t))/dt=0$, Eq.(25) can be rewritten as $\dot{f}(t)=\langle\tau(t),\dot{\eta}(t)\rangle+H(t)$ with

$$H(t)=-\frac{1}{2}\langle\tau(t),\dot{\mathcal{L}}_0(t)\tau(t)\rangle. \tag{26}$$

Integrating on the time interval $[a,a+T]$, we obtain

$$f(a+T)-f(a)=\int_a^{a+T}\langle\tau(t),\dot{\eta}(t)\rangle\,dt+\int_a^{a+T}H(t)\,dt. \tag{27}$$

In the right-hand side of (27), the first term is an irreversible contribution associated with the plastic dissipation whereas the second term is a reversible contribution associated with the time fluctuations of the elastic moduli. In the following, we bound those two terms separately, starting with the irreversible contribution $-\int_a^{a+T}\langle\tau(t),\dot{\eta}(t)\rangle\,dt$. To that purpose, we use a reasoning that is quite similar to that used in [23] for perfect plasticity.

We first note from (12) that $\dot{\eta}(t)=-\pi\Phi'(\rho(t)+\sigma^E(t),t)$. Since Φ is convex, positive and vanishes in $\mathcal{K}_0(t)$, we have

$$0\leq\Phi(\rho(t)+\sigma^E(t),t)\leq\langle\dot{\eta}(t),\rho'(t)-\rho(t)\rangle \text{ for all } \rho'\in\mathcal{K}(t)\cap\mathcal{H}. \tag{28}$$

The strong Melan's condition implies that $\rho^*(t)-r\dot{\eta}/\|\dot{\eta}\|$ is in $\mathcal{K}(t)\cap\mathcal{H}$. Hence (28) gives

$$0\leq\langle\dot{\eta}(t),\rho^*(t)-r\frac{\dot{\eta}}{\|\dot{\eta}\|}-\rho(t)\rangle=\langle\dot{\eta}(t),\rho^*(t)-\rho(t)\rangle-r\|\dot{\eta}(t)\|,$$

i.e. $\langle\dot{\eta}(t),\rho(t)-\rho^*(t)\rangle\leq-r\|\dot{\eta}(t)\|$. Hence

$$\int_a^{a+T}\langle\tau(t),\dot{\eta}(t)\rangle\,dt\leq-r(\inf\mu)\int_a^{a+T}\|\dot{\eta}\|\,dt. \tag{29}$$

Bounding the reversible contribution $\int_a^{a+T}H(t)\,dt$ requires a little more effort. We have

$$\tau(t)=\mathcal{L}_0^{-1}(t)(\eta(t)-\eta(a)+s) \tag{30}$$

where

$$s=\mathcal{L}(a)(\rho(a)-\rho^*(a)) \tag{31}$$

is independent on t. Substituting the expression (30) in (26) and using the fact that $\mathcal{L}_0(t)$ is self-adjoint, we find

$$H(t) = -\frac{1}{2}\langle(\eta(t)-\eta(a)+s), \mathcal{L}_0^{-1}(t)\dot{\mathcal{L}}_0(t)\mathcal{L}_0^{-1}(t)(\eta(t)-\eta(a)+s)\rangle.$$

Observing that $\mathcal{L}_0^{-1}(t)\dot{\mathcal{L}}_0(t)\mathcal{L}_0^{-1}(t) = -\dot{\mathcal{M}}_0(t)$ yields

$$H(t) = \frac{1}{2}\langle(\eta(t)-\eta(a)+s), \dot{\mathcal{M}}_0(t)(\eta(t)-\eta(a)+s)\rangle. \tag{32}$$

Since s does not depend on t and $\mathcal{M}_0(t)$ is $T-$periodic, the integration of (32) on the time interval $[a, a+T]$ gives

$$\int_a^{a+T} H(t)dt = \frac{1}{2}\int_a^{a+T}\langle \eta(t)-\eta(a), \dot{\mathcal{M}}_0(t)(\eta(t)-\eta(a)+2s)\rangle dt. \tag{33}$$

Using now the Cauchy-Schwarz inequality and the definition of the norm operator, we obtain

$$\langle \eta(t)-\eta(a), \dot{\mathcal{M}}_0(t)(\eta(t)-\eta(a)+2s)\rangle \leq \|\eta(t)-\eta(a)\| \cdot |||\dot{\mathcal{M}}_0(t)||| \cdot \|\eta(t)-\eta(a)+2s\|.$$

Since $\eta(t)-\eta(a)+2s = \eta(t)+\eta(a)-2\mathcal{L}(a)\rho^*(a)$, we have

$$\|\eta(t)-\eta(a)+2s\| \leq \|\eta(t)\| + \|\eta(a)\| + 2\|\mathcal{L}(a)\rho^*(a)\|.$$

Lemma 1 gives $\|\eta(t)\| \leq M'$ and $\|\eta(a)\| \leq M'$. Since $\rho^*(a) \in \mathcal{K}(a)\cap\mathcal{H}$, we have $\|\mathcal{L}(a)\rho^*(a)\| \leq CM$ as a consequence of (16-17). Hence

$$\|\eta(t)-\eta(a)+2s\| \leq 2M' + 2CM \leq 4M'. \tag{34}$$

We also have, for $t \in [a, a+T]$,

$$\|\eta(t)-\eta(a)\| = \|\int_a^t \dot{\eta}(t')dt'\| \leq \int_a^t \|\dot{\eta}(t')\|dt' \leq \int_a^{a+T} \|\dot{\eta}(t')\|dt'. \tag{35}$$

Substituting (34-35) in (33) and using the definition (18) of $\gamma(0,T)$, we obtain

$$\int_a^{a+T} H(t)dt \leq 2M'\gamma(0,T)\int_a^{a+T}\|\dot{\eta}(t)\|dt \tag{36}$$

Replacing (29) and (36) in (24) gives the desired result. □

3.3 Proof of the theorem

We are now in a position to prove Theorem 1. By condition (i), one can pick $M' > CM$ such that $\frac{\gamma(0,T)}{\inf \mu} < \frac{r}{2M'}$. Using Lemma 1 we such M', there exists t_0 such that $\|\eta(t)\| \leq M'$ for $t \geq t_0$. Let now $N_0 \in \mathbb{N}$ be such that $N_0 T \geq t_0$. For $i \geq N_0$, Lemma 2 gives

$$f((i+1)T) - f(i) \leq -m \int_{iT}^{(i+1)T} \|\dot{\eta}(t)\| \, dt$$

where $m = r(\inf \mu) - 2M'\gamma(0,T)$ is non-negative by (ii). Summing over $i = N_0, \cdots, N$ and recalling that f is positive, we find

$$\int_{N_0 T}^{NT} \|\dot{\eta}(t)\| \, dt \leq \frac{1}{m} f(N_0 T).$$

Taking the limit $N \to \infty$ shows that the integral $\int_0^S \|\dot{\eta}(t)\| dt$ converges as $S \longrightarrow +\infty$. Since $\mathcal{E}$ is a Hilbert space, it follows that $\eta(t)$ also converges towards a limit η_∞ as $t \longrightarrow +\infty$. That limit η_∞ is in $\mathcal{H}$ because $\mathcal{H}$ is closed in $\mathcal{E}$. Setting $\rho_\infty(t) = \mathcal{M}(t)\eta_\infty$ and recalling that $\mathcal{M}(t)$ is bounded, we have $\rho(t) - \rho_\infty(t) \longrightarrow 0$ as $t \to \infty$. We now check that $\rho_\infty(t)$ is an elastic solution, i.e. satisfies (13). The definition of $\rho_\infty(t)$ rightly gives $d(\mathcal{L}(t)\rho_\infty(t))/dt = 0$, but the fact that $\rho_\infty(t) \in \mathcal{K}(t)$ calls for some justification. Noting by (12) that $\dot{\eta}(t) = -\alpha(\rho(t) - \pi P_{\mathcal{K}(t)}\rho(t))$, we have

$$\|\rho_\infty(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\| \leq \frac{1}{\alpha}\|\dot{\eta}(t)\| + \|\rho(t) - \rho_\infty(t)\| + \|\pi P_{\mathcal{K}(t)}\rho(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\|.$$

It is a classical result that the projection operator on a closed convex set is a contractive mapping [25], hence $\|\pi P_{\mathcal{K}(t)}\rho(t) - \pi P_{\mathcal{K}(t)}\rho_\infty\| \leq \|\rho(t) - \rho_\infty(t)\|$ and

$$\|\rho_\infty(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\| \leq \frac{1}{\alpha}\|\dot{\eta}(t)\| + 2\|\rho(t) - \rho_\infty(t)\|.$$

Recall that $\rho(t) - \rho_\infty(t) = \mathcal{M}(t)(\eta(t) - \eta_\infty)$. Since $\mathcal{M}(t)$ is bounded, there is a constant K such that $\|\rho(t) - \rho_\infty(t)\| \leq K\|\eta(t) - \eta_\infty\|$. Therefore,

$$\|\rho_\infty(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\| \leq \frac{1}{\alpha}\|\dot{\eta}(t)\| + 2K\|\eta(t) - \eta_\infty(t)\|.$$

Integrating on the time interval $[iT, (i+1)T]$ and observing that $\rho_\infty(t)$ and $\mathcal{K}(t)$ are $T-$periodic, we obtain

$$\int_0^T \|\rho_\infty(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\| \, dt \leq \frac{1}{\alpha}\int_{iT}^{(i+1)T} \|\dot{\eta}(t)\| \, dt + 2K \int_{iT}^{(i+1)T} \|\eta(t) - \eta_\infty\| \, dt.$$

Since $\int_0^\infty \|\dot{\eta}(t)\| \, dt < \infty$, we have

$$\int_{iT}^{(i+1)T} \|\dot{\eta}(t)\| \, dt \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

Moreover, since $\eta(t)$ converges towards η_∞, we also have

$$\int_{iT}^{(i+1)T} \|\eta(t) - \eta_\infty\| \, dt \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

It follows that $\|\rho_\infty(t) - \pi P_{\mathcal{K}(t)}\rho_\infty(t)\| = 0$ on $[0,T]$, i.e. $\rho_\infty(t) = \pi P_{\mathcal{K}(t)}\rho_\infty(t)$ on $[0,T]$. Therefore, we have $P_{\mathcal{K}(t)}\rho_\infty(t) = \rho_\infty(t) + q$ with $q \in \mathcal{H}^\perp$. By the definition of the projection it follows that

$$\langle \rho_\infty(t) - P_{\mathcal{K}(t)}\rho_\infty(t), \rho' - P_{\mathcal{K}(t)}\rho_\infty \rangle \leq 0$$

for any $\rho' \in \mathcal{K}(t)$. In particular, choosing $\rho' = \rho^*(t) \in \mathcal{K}(t) \cap \mathcal{H}$ we find

$$-\langle q, \rho^*(t) - \rho_\infty(t) - q \rangle = \|q\|^2 \leq 0$$

hence $q = 0$, i.e. $P_{\mathcal{K}(t)}\rho_\infty(t) = \rho_\infty(t)$ or equivalently $\rho_\infty(t) \in \mathcal{K}(t)$. This completes the proof that $\rho_\infty(t)$ is an elastic solution. Since $\rho_\infty(t) - \rho(t) \longrightarrow 0$ as $t \to \infty$, the convergence of $\rho(t)$ towards an elastic solution is obtained.

Remark: It is interesting to compare Theorem 1 with the analog result obtained in [23] for elastic-perfectly plastic materials. In [23], a condition analog to (ii) was formulated, with the factor 2 in (ii) replaced by a factor 3. That factor 3 results from Eq. (46) in [23], which – using the present notations – states that $\|\eta(t) - \eta(a) + 2s\| \leq 6CM$. Using a reasoning similar to that used in Eq. (34), one can actually observe that $\|\eta(t) - \eta(a) + 2s\| \leq 4CM$, which leads to an improved factor 2 instead of a factor 3.

There is also a more subtle difference between Theorem 1 above and the results in [23], regarding the definition of the constant M. In Theorem 1, M is a bound on $\pi\mathcal{K}(t)$ whereas in [23] M is a bound on $\mathcal{K}(t) \cap \mathcal{H}$. We have $\mathcal{K}(t) \cap \mathcal{H} \subset \pi\mathcal{K}(t)$ but the inclusion is generally strict.

4 Illustrative example

As an illustration of Theorem 1, consider the problem of an elastic-viscoplastic plate under cyclic thermal loading. The plate is stress-free in the e_3 direction and is in frictionless contact with rigid walls in the (e_1, e_2) directions (Fig. 1). Length units are chosen such that the plate has unit volume. The constitutive material is homogeneous and the dissipation potential ϕ of the Perzyna type

$$\phi(\sigma) = \frac{1}{\alpha}\langle J_2 - \sigma_y \rangle_+^2 \tag{37}$$

where α is a viscosity parameter, σ_y is the yield strength, $J_2 = \sqrt{3/2}|\sigma - (\mathrm{tr}\,\sigma/3)I|$ is the second invariant of the deviatoric stress, and $\langle x \rangle_+ = \max(0,x)$ denotes the positive part of a scalar x. The potential (37) can be put in the format (3) by setting $\mathscr{C}(x,t) = \{\sigma : J_2 \leq \sigma_y\}$.

Because of the rigid walls in the e_1 and e_2 directions, thermal dilatation may result in high compressive stress and plastic flow. In the following we are interested in bounding the temperature fluctuations θ for which shakedown occurs.

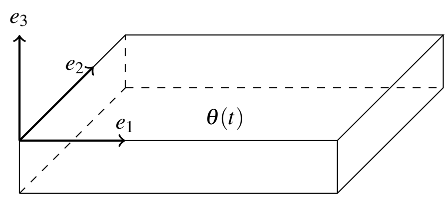

Fig. 1 An elastic-viscoplastic plate under a cyclic temperature $\theta(t)$. The plate is constrained in the e_1 and e_2 directions.

4.1 Temperature-independent elastic moduli

Using the plane stress assumption, we consider a 2-dimensional model of the problem. If the imposed temperature field θ as well as the initial state $\varepsilon^p(t=0)$ are uniform – which is assumed in the following – then the fields ε^p, ε and σ remain uniform at all time. The space $\mathcal{E}$ is thus chosen as the 3-dimensional space of tensors σ with a matrix representation of the form

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{38}$$

in the basis (e_1, e_2, e_3). The space $\mathcal{H}$ of residual stresses is the subspace of $\mathcal{E}$ constituted by uniform fields ρ of the form

$$\rho = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

The purely elastic stress response $\sigma^E(t)$ of the plate is given by

$$\sigma^E = \begin{pmatrix} f^E(t) & 0 & 0 \\ 0 & f^E(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{40}$$

where $f^E(t) = -E(t)\varepsilon^\theta(t)/(1-\nu(t))$. It can easily be verified that

$$\mathcal{K}_0 = \{\sigma \in \mathcal{E} : \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 6\sigma_{12}^2 \le \sigma_y^2\}, \tag{41}$$

$$\pi\mathcal{K}_0 = \{\rho \in \mathcal{H} : \rho_1^2 + \rho_2^2 - \rho_1\rho_2 \le \sigma_y^2\}. \tag{42}$$

Let $(\mathbf{E}_1, \mathbf{E}_2)$ be the orthonormal basis of $\mathcal{H}$ defined by $\mathbf{E}_1 = \mathrm{diag}(1,0,0)$ and $\mathbf{E}_2 = \mathrm{diag}(0,1,0)$. The set $\pi\mathcal{K}_0$ in (42) is a solid ellipsoid with axes $\mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{E}_1 - \mathbf{E}_2$. The set $\pi\mathcal{K}(t) = \pi(\mathcal{K}_0 - \sigma^E(t))$ is obtained by a time-dependent translation of $\pi\mathcal{K}_0$ in the $\mathbf{E}_1 + \mathbf{E}_2$ direction (Fig. 2).

Setting $\sigma^\theta = \sup_t |\sigma^E(t)|$, it can easily be verified that $\cap_t \pi\mathcal{K}(t)$ is non empty as long as $\sigma^\theta \le \sigma_y$. More precisely, $\cap_t \pi\mathcal{K}(t)$ contains $B(0,r) \cap \mathcal{H}$ where $B(0,r)$ is the ball centered at the origin with a radius r given by

$$r = \begin{cases} \sigma_y \sqrt{\dfrac{2}{3} - \left(\dfrac{\sigma^\theta}{\sigma_y}\right)^2} & \text{if } 0 \le \dfrac{\sigma^\theta}{\sigma_y} \le \dfrac{2}{3}, \\ \sigma_y \sqrt{2}(1 - \dfrac{\sigma^\theta}{\sigma_y}) & \text{if } \frac{2}{3} \le \dfrac{\sigma^\theta}{\sigma_y} \le 1. \end{cases} \tag{43}$$

Since $\rho = 0$ is an elastic solution to the evolution problem, it can be deduced from Theorem 1 (or from the standard form of Melan's theorem) that shakedown occurs if $\sigma^\theta \le \sigma_y$.

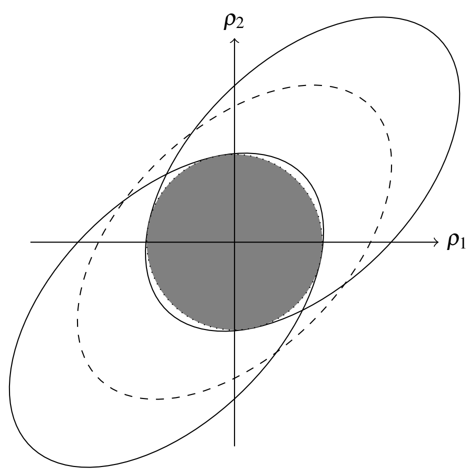

Fig. 2 The set $\pi\mathcal{K}(t)$ is a solid ellipsoid obtained by translation of the ellipsoid $\pi\mathcal{K}_0$ (shown in dotted lines) in the (1,1) direction. Shown in solid lines are the extreme locations of $\pi\mathcal{K}(t)$, corresponding to $f^E(t) = \pm\sigma^\theta$. The filled ball centered at the origin is included in $\pi\mathcal{K}(t)$ for all t.

4.2 Temperature-dependent elastic moduli

Let us now use Theorem 1 to estimate the shakedown limit in the case of non constant elastic moduli. Applying Theorem 1 requires to evaluate the constants M, C in (16-17) as well as the scalar $\gamma(0,T)$ defined in (18).

It can be verified from (42) that any σ in $\pi\mathcal{K}_0$ satisfies $\|\sigma\| \leq \sqrt{2}\sigma_y$. Hence any σ in $\pi\mathcal{K}(t)$ satisfies

$$\|\sigma\| \leq \sqrt{2}\sigma_y + \|\sigma^E(t)\| \leq \sqrt{2}(\sigma_y + \sigma^\theta).$$

The constant M in (16) can thus be chosen as

$$M = \sqrt{2}(\sigma_y + \sigma^\theta). \tag{44}$$

Recall that $\mathcal{L}(t)\rho$ is defined as the projection of $L(t) : \rho$ on $\mathcal{H}$. In the present case, we have

$$\mathcal{L}(t)\rho = \frac{1}{E(t)} \begin{pmatrix} \rho_1 - \nu\rho_2 & 0 & 0 \\ 0 & \rho_2 - \nu\rho_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the matrix representation of $\mathcal{L}$ in the basis $(\mathbf{E}_1, \mathbf{E}_2)$ is given by

$$\mathcal{L}(t) = \frac{1}{E(t)} \begin{pmatrix} 1 & -\nu \\ -\nu & 1 \end{pmatrix}.$$

The operator $\mathcal{L}(t)$ being symmetric, its norm operator $|||\mathcal{L}(t)|||$ is equal to $\max_i |\lambda_i|$ where $\{\lambda_i\}$ are the eigenvalues of $\mathcal{L}(t)$. A simple calculation gives

$$|||\mathcal{L}(t)||| = \frac{1+\nu(t)}{E(t)}. \tag{45}$$

Set

$$\nu_{min} = \inf_t \nu(t),\ \nu_{max} = \sup_t \nu(t),\ E_{min} = \inf_t E(t),\ E_{max} = \sup_t E(t).$$

From (45) we have $|||\mathcal{L}(t)||| \leq \frac{1+\nu_{max}}{E_{min}}$ for all t. Hence the constant C in (17) can be chosen as

$$C = \frac{1+\nu_{max}}{E_{min}}. \tag{46}$$

In order to calculate $\gamma(0,T)$, we note that

$$\mathcal{M}(t) = \mathcal{L}^{-1}(t) = \frac{E(t)}{1-\nu^2(t)} \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix}.$$

Choosing $\mu(t) = 1/\operatorname{tr}\mathcal{M}(t)$ as suggested in [23], we get

$$\mu(t) = \frac{1-\nu^2(t)}{2E(t)},\ \mathcal{M}_0(t) = \frac{1}{2}\begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix},\ \dot{\mathcal{M}}_0(t) = \frac{1}{2}\begin{pmatrix} 0 & \dot{\nu} \\ \dot{\nu} & 0 \end{pmatrix}.$$

It follows that $|||\dot{\mathcal{M}}_0(t)||| = \frac{1}{2}|\dot{\nu}|$ and

$$\gamma(0,T) = \frac{1}{2}\int_0^T |\dot{\nu}|dt.$$

Using the values of M and C defined in (44) and (46) respectively, we obtain from Theorem 1 that shakedown occurs if

$$\frac{1+\nu_{max}}{E_{min}} \frac{\int_0^T |\dot{\nu}|dt}{\inf_t \frac{1-\nu^2(t)}{E(t)}} \leq \frac{r}{\sqrt{2}(\sigma_y + \sigma^\theta)} \tag{47}$$

where r is the function of σ^θ defined in (43). The left-hand side of (47) is a function of the fluctuations of the elastic moduli, whereas the right-hand side is a function of the loading σ^θ. Observing that $(1-\nu^2(t))/E(t) \geq (1-\nu_{max}^2)/E_{max}$, it can be seen that a sufficient condition for Eq. (47) to be satisfied is

$$\frac{1}{1-\nu_{max}} \frac{E_{max}}{E_{min}} \int_0^T |\dot{\nu}|dt \leq \frac{r}{\sqrt{2}(\sigma_y + \sigma^\theta)}. \tag{48}$$

Assume that $\dot{\nu}$ vanishes only when $\nu(t) = \nu_{min}$ or $\nu(t) = \nu_{max}$, i.e. $\nu(t)$ grows monotonically from ν_{min} to ν_{max} and then decreases monotonically from ν_{max} to ν_{min}, in a periodic fashion. In such case, we have

$$\int_0^T |\dot{\nu}|dt = 2(\nu_{max} - \nu_{min})$$

and the inequality (48) reduces to

$$\frac{(\nu_{max} - \nu_{min})}{1-\nu_{max}} \frac{E_{max}}{E_{min}} \leq \frac{r}{2\sqrt{2}(\sigma_y + \sigma^\theta)}. \tag{49}$$

For fixed values of E_{max}/E_{min}, the shakedown limit σ^θ_{SD} is defined at the largest value of σ^θ that satisfies (49). The shakedown limit σ^θ_{SD} is plotted in Fig. 3 as a function of E_{max}/E_{min}, for several values of $\nu_{max} - \nu_{min}$. The maximum value ν_{max} is set to 0.3 for all the curves in Fig. 3. First consider the case $\nu_{max} - \nu_{min} = 0$: Whatever the value of E_{max}/E_{min}, the shakedown limit is equal to σ_y and thus coincides with the value obtained in Sect. 4.1 for temperature-independent elastic moduli. When $\nu_{max} - \nu_{min}$ increases, the shakedown limit decreases rapidily. For instance, the shakedown limit is approximatively equal to $0.6\sigma_y$ in the case of a 20% variation of E and ν around nominal values $(E_0, \nu_0 = 0.3)$. For common metals, the Young modulus E decreases with the temperature, while the Poisson ratio increases [27]. The Young modulus is more sensitive to temperature variations than the Poisson ratio. To put things in perspective, a 20% variation of the Young modulus E in steels would typically correspond to temperatures fluctuating between $0\,^\circ$C and $500\,^\circ$C [28].

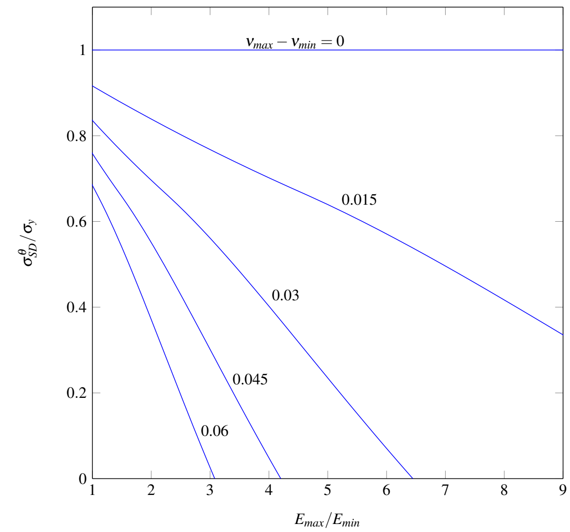

Fig. 3 Shakedown limit as a function of E_{max}/E_{min}, plotted for several values of $\nu_{max} - \nu_{min}$. Case $\nu_{max} = 0.3$.

4.3 Incremental analysis

As a further illustration of Theorem 1, we use incremental analysis to solve (12) for given loading history and fluctuations of the elastic moduli. The functions $f^E(t)$, $E(t)$ and $\nu(t)$ are taken as $f^E(t) = \sigma^\theta \sin \omega t$, $E(t) = E_0(1 - 0.2\sin^2 \omega t)$, $\nu(t) = \nu_0(1 + 0.2\sin^2 \omega t)$ with $E_0/\sigma_y = 10$ and $\nu_0 = 0.3$. For such parameters, the shakedown limit σ^θ_{SD} as provided by Eq. (49) is approximatively equal to $0.6\sigma_y$. The differential equation (12) is solved numerically using a Runge-Kutta scheme with variable step size. In Fig. 4 is shown the evolution of the plastic strain in the case $\sigma^\theta = 0.5\sigma_y$, i.e. below the shakedown limit. The initial state is taken as $\rho_1(t = 0) = \sigma_y$, $\rho_2(t = 0) = 2\sigma_y$. In accordance with the results of Sect. 4.2, the plastic strain converges towards a limit as $t \to \infty$. Theorem 1 ensures that such behavior occurs for *all* inital states $\rho_1(t = 0)$, $\rho_2(t = 0)$. In Fig. 5 is plotted the evolution of $\|\varepsilon^p(t)\|$ (solid line). A fast decrease is observed for small time: This is the meaning of Lemma 1 introduced in Sect. 3.1. In dashed line is plotted the exponental upper bound on $\|\varepsilon^p(t)\|$ that is deduced from Eq. (22) .

5 Concluding remarks

The shakedown theorem presented in this chapter gives a sufficient condition for the evolution to become elastic in the large time limit, whatever the initial state is. Loosely speaking, that theorem states that if the loading is not too large (in the sense of condition (i)) *and* the time-fluctuations of the elastic moduli are not too large (in the sense of condition (ii)), then shakedown occurs immaterial of the initial state.

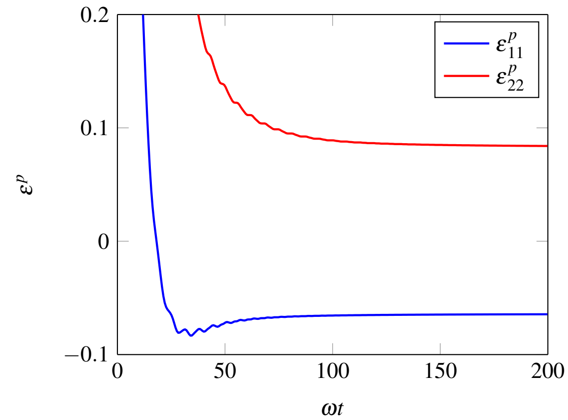

Fig. 4 Evolution of the plastic strain for a loading below the shakedown limit. Case $\rho_1(t = 0) = \sigma_y$, $\rho_2(t = 0) = 2\sigma_y$.

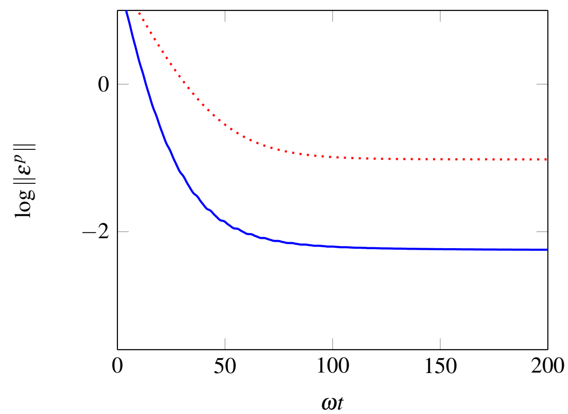

Fig. 5 Evolution of $\|\varepsilon^p(t)\|$ (blue solid line) and comparison with the exponential upper bound provided by Eq. (22) (red dotted line).

We emphasize again that setting a restriction on the time-fluctuations of the elastic moduli is essential to ensure that shakedown occurs in a path-independent fashion.

It can be observed that Theorem 1 is independent on the viscosity parameter α. This is consistent with the case of constant elastic moduli: the shakedown behavior is essentially determined by the elasticity domain of the material (as long as the dissipation ϕ satisfies standard assumptions such as convexity).

For the sake of simplicity, the viscosity parameter α has been assumed to be independent on (x,t) but there is no difficulty in extending Theorem 1 to non constant viscosities. It could be interesting to investigate whether Theorem 1 could be extended to other viscoplastic potentials than those of the form (3).

In practice, the shakedown theorem that has been presented could be useful for the fatigue design of structures submitted to large temperature variations. As illustrated in Sect. 4, that theorem gives lower bounds on the shakedown limit. It would be interesting to investigate whether the kinematic shakedown theorem of [3] could be extended to the case of temperature-dependent elastic moduli, so as to obtain upper bounds.

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