A Finite-Volume discretization of viscoelastic Saint-Venant equations for FENE-P fluids

Sébastien Boyaval

A Finite-Volume discretization of viscoelastic Saint-Venant equations for FENE-P fluids

Sébastien Boyaval

Abstract Saint-Venant equations can be generalized to account for a viscoelastic rheology in shallow flows. A Finite-Volume discretization for the 1D Saint-Venant system generalized to Upper-Convected Maxwell (UCM) fluids was proposed in [Bouchut & Boyaval, 2013], which preserved a physically-natural stability property (i.e. free-energy dissipation) of the full system. It invoked a relaxation scheme of Suliciu type for the numerical computation of approximate solution to Riemann problems. Here, the approach is extended to the 1D Saint-Venant system generalized to the finitely-extensible nonlinear elastic fluids of Peterlin (FENE-P). We are currently not able to ensure all stability conditions a priori, but numerical simulations went smoothly in a practically useful range of parameters.

Key words: Saint-Venant equations, FENE-P viscoelastic fluids, Finite-Volume, simple Riemann solver, Suliciu relaxation scheme

MSC (2010)**:** 65M08, 65N08, 35Q30

1 Introduction

Saint-Venant equations standardly model shallow free-surface gravity flows and can be generalized to account for the viscoelastic rheology of non-Newtonian fluids [6], Upper-Convected Maxwell (UCM) fluids in particular [5]. Here, we consider a generalized Saint-Venant (gSV) system for *finitely-extensible nonlinear elastic* fluids with Peterlin closure (FENE-P fluids) in Cartesian coordinates

---

Sébastien Boyaval
Laboratoire d'hydraulique Saint-Venant (Ecole des Ponts ParisTech – EDF R& D – CEREMA) Université Paris-Est, EDF'lab 6 quai Watier 78401 Chatou Cedex France, & INRIA Paris MATHERIALS e-mail: sebastien.boyaval@enpc.fr

$$\partial_t h + \partial_x(hu) = 0 \quad (1)$$

$$\partial_t(hu) + \partial_x\left(hu^2 + gh^2/2 + hN\right) = 0 \quad (2)$$

$$\lambda\left(\partial_t \sigma_{xx} + u\partial_x \sigma_{xx} + 2(\zeta - 1)\sigma_{xx}\partial_x u\right) = 1 - \sigma_{xx}/(1 - (\sigma_{zz} + \sigma_{xx})/\ell) \quad (3)$$

$$\lambda\left(\partial_t \sigma_{zz} + u\partial_x \sigma_{zz} + 2(1 - \zeta)\sigma_{zz}\partial_x u\right) = 1 - \sigma_{zz}/(1 - (\sigma_{zz} + \sigma_{xx})/\ell) \quad (4)$$

for 1D $\mathbf{e}_y$-translation invariant flow along $\mathbf{e}_x$ under a uniform gravity field $-g\mathbf{e}_z$ with

- mean flow depth $h(t,x) > 0$ (in case of a non-rugous flat bottom),
- mean flow velocity $u(t,x)$ (for *uniform* cross sections), and
- a normal-stress difference $N = G(\sigma_{zz} - \sigma_{xx})/(1 - (\sigma_{zz} + \sigma_{xx})/\ell)$ given by conformation variables $\sigma_{zz}, \sigma_{xx} > 0$ constrained by $0 < \sigma_{zz} + \sigma_{xx} < \ell$, a relaxation time $\lambda \geq 0$ and an elasticity modulus $G > 0$.

Note that (1-2-3-4) formally reduces to the standard viscous Saint-Venant system with viscosity $\nu \equiv 2\lambda G \geq 0$ when $\ell \to \infty$, $\lambda \to 0$ and $G\lambda < \infty$. Moreover we have used the quite general Gordon-Schowalter derivatives with slip parameter $\zeta \in [0, \frac{1}{2}]$ constrained by the hyperbolicity of the system (1-2-3-4). (This follows after an easy computation similar to [8].)

In this work, we discuss a Finite-Volume method to solve (numerically) the Cauchy problem for the nonlinear hyperbolic 1D system (1-2-3-4). Standardly, we need to consider *weak* solutions (in fact, to (6-7-8-9), see below) plus *admissibility* constraints that are physically-meaningful dissipation rules formalizing the thermodynamics second principle close to an equilibrium [9]. Here, we consider the *inequality* associated with the companion conservation law for the *free-energy*

$$F = h\left(\frac{u^2}{2} + \frac{gh}{2} - \frac{G}{2(1-\zeta)}\left(\ell \log\left((\ell - (\sigma_{xx} + \sigma_{zz}))/(\ell - 2)\right) + \log(\sigma_{xx}\sigma_{zz})\right)\right)$$

that is, on denoting the impulse by $P = gh^2/2 + hN$,

$$-\frac{Gh}{2(1-\zeta)\lambda}\left(\sigma_{xx}^{-1}\left(1 - \frac{\sigma_{xx}}{1 - (\sigma_{zz} + \sigma_{xx})/\ell}\right)^2 + \sigma_{zz}^{-1}\left(1 - \frac{\sigma_{zz}}{1 - (\sigma_{zz} + \sigma_{xx})/\ell}\right)^2\right)$$
$$=: D \geq \partial_t F + \partial_x\left(u(F + P)\right) \quad (5)$$

where the left-hand-side is obviously non-positive on the admissibility domain

$$\mathscr{U}^\ell := \{0 < h, 0 < \sigma_{xx}, 0 < \sigma_{zz}, \sigma_{xx} + \sigma_{zz} < \ell\}.$$

Note that we do not consider the vacuum state $h = 0$ as admissible here, see [8].

2 Finite-Volume discretization of FENE-P/Saint-Venant

Piecewise-constant approximate solutions to the Cauchy problem on $(t,x) \in [0,T) \times \mathbb{R}$ for the gSV system can be defined by a Finite-Volume (FV) method. With a view to preserving $\mathscr{U}^\ell$ and the dissipation (5) after discretization by a FV method, we choose $q = (h, hu, h\sigma_{xx}, h\sigma_{zz})$ as discretization variable. Indeed, the free-energy functional F is *convex* on the convex domain $\mathscr{U}^\ell \ni q$ (this follows after an easy computation from [4, Lemma 1.3]) while it is not convex in the variable $(h, hu, h\Pi, h\Sigma)$ whatever smooth invertible functions ϖ, ς are used for the reformulation of gSV

$$\partial_t h + \partial_x (hu) = 0 \tag{6}$$

$$\partial_t (hu) + \partial_x \left(hu^2 + \frac{gh^2}{2} + hN \right) = 0 \tag{7}$$

$$\partial_t (h\Pi) + \partial_x (hu\Pi) = \frac{h^{3-2\zeta} \varpi'(\sigma_{xx} h^{2(1-\zeta)})}{\lambda} \left(1 - \frac{\sigma_{xx}}{1 - \frac{\sigma_{zz}+\sigma_{xx}}{\ell}} \right) \tag{8}$$

$$\partial_t (h\Sigma) + \partial_x (hu\Sigma) = \frac{h^{2\zeta-1} \varsigma'(\sigma_{zz} h^{2(\zeta-1)})}{\lambda} \left(1 - \frac{\sigma_{zz}}{1 - \frac{\sigma_{zz}+\sigma_{xx}}{\ell}} \right) \tag{9}$$

with $\Pi = \varpi(\sigma_{xx} h^{2(1-\zeta)})$, $\Sigma = \varsigma(\sigma_{zz} h^{2(\zeta-1)})$ (computations are similar to [5, Appendix]). In the sequel, we therefore discretize a quasilinear system with source

$$\partial_t q + A(q)\partial_x q = S(q)\,, \tag{10}$$

which we recall is not ambiguous here (for those discontinuous solutions built using a Riemann solver, at least) thanks to the dissipation rule (5), see [11, 2, 8].

2.1 *Splitting-in-time*

In cell $(x_{i-1/2}, x_{i+1/2})$, $i \in \mathbb{Z}$, with volume $\Delta x_i = x_{i+1/2} - x_{i-1/2} > 0$ and center $x_i = (x_{i-1/2} + x_{i+1/2})/2$, we approximate q solution to (10) on $\mathbb{R}_{\geq 0} \times \mathbb{R} \ni (t,x)$ by

$$q_i^{n+1} \approx \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} q(t,x) dx,\ i \in \mathbb{Z}, t \in (t^n, t^{n+1}]$$

on a time grid $0 = t^0 < t^1 < \ldots < t^n < t^{n+1} < \ldots < t^N = T$ where $\Delta t^n = |t^{n+1} - t^n|$ will be chosen small enough compared with $\Delta x = \sup_{i \in \mathbb{Z}} \Delta x_i < \infty$ to ensure stability.

More precisely, having in mind the numerical approximation of a (well-posed) Cauchy problem for (10) on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with initial condition $q(t \to 0^+) = q^0 \in L^\infty(\mathbb{R})$, and therefore starting from approximations $q_i^0 \approx \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} q^0(x) dx$, $i \in \mathbb{Z}$, we shall define the cell values q_i^n in two steps for each $n = 1, \ldots, N$:
(i) an approximate solution to the *homogeneous* gSV system (i.e. without the source

term S) on $[t^n, t^{n+1})$ is first computed by an explicit three-point scheme

$$q_i^{n+1/2} = q_i^n - \frac{\Delta t^n}{\Delta x_i}\left(F_l(q_i^n, q_{i+1}^n) - F_r(q_{i-1}^n, q_i^n)\right), \tag{11}$$

(ii) an approximate solution to the full gSV system on $(t^n, t^{n+1}]$ is next computed as

$$q_i^{n+1} = q_i^{n+1/2} + \Delta t^n S(q_i^{n+1}). \tag{12}$$

Then, the scheme is consistent with weak solutions of (1–2) equiv. (6–7)

$$q_i^{n+1} = q_i^n - \frac{\Delta t^n}{\Delta x_i}\left(F_l(q_i^n, q_{i+1}^n) - F_r(q_{i-1}^n, q_i^n)\right) + \Delta t^n S(q_i^{n+1}) \tag{13}$$

provided the two first flux components for the conservative part (h, hu) of the variable q (actually solutions to conservation laws) are conservative $F_{l,h} = F_{r,h} := F_h$, $F_{l,hu} = F_{r,hu} := F_{hu}$ and consistent $F_h(q,q) = hu|_q$, $F_{hu}(q,q) = (hu^2 + gh^2/2 + hN)|_q$ as usual, and with the conservative interpretation (8–9) of (3–4) insofar as we next define F_l and F_r using a *simple* approximate Riemann solver [10] for (6–7–8–9).

Moreover, with a view to preserving $\mathscr{U}^\ell$ and a discrete version of (5)

$$F(q_i^{n+1/2}) - F(q_i^n) + \frac{\Delta t^n}{\Delta x_i}\left(G(q_i^n, q_{i+1}^n) - G(q_{i-1}^n, q_i^n))\right) \leq 0 \tag{14}$$

for a numerical free-energy flux function consistent with $G(q,q) = u(F+P)|_q$ in (5), in the sequel, we shall discuss the relaxation technique introduced by Suliciu as simple Riemann solver in step (i), because it proved satisfying for other close systems [3, 4, 5] equipped with an "entropy" convex in the discretization variable like F here. In the end, for the full scheme (13), a consistent free-energy dissipation

$$F(q_i^{n+1}) - F(q_i^n) + \frac{\Delta t^n}{\Delta x_i}\left(G(q_i^n, q_{i+1}^n) - G(q_{i-1}^n, q_i^n))\right) \leq \Delta t^n D(q_i^{n+1}) \tag{15}$$

then holds insofar $h_i^{n+1/2} = h_i^{n+1}$, $u_i^{n+1/2} = u_i^{n+1}$ and the convexity of F imply

$$F(q_i^{n+1}) - F(q_i^{n+1/2}) \leq \Delta t^n D(q_i^{n+1}) \leq 0. \tag{16}$$

Proof. On noting $h_i^{n+1/2} = h_i^{n+1}$, $u_i^{n+1/2} = u_i^{n+1}$ it suffices to show that

$$\lambda\left(\sigma_{xx,i}^{n+1} - \sigma_{xx,i}^n\right)/\Delta t^n = 1 - \sigma_{xx,i}^{n+1}/(1 - (\sigma_{zz,i}^{n+1} + \sigma_{xx,i}^{n+1})/\ell)$$
$$\lambda\left(\sigma_{zz,i}^{n+1} - \sigma_{zz,i}^n\right)/\Delta t^n = 1 - \sigma_{zz,i}^{n+1}/(1 - (\sigma_{zz,i}^{n+1} + \sigma_{xx,i}^{n+1})/\ell)$$

imply (16). Now, this is obvious, on noting the convexity of $F|_{h,u}$ in $(\sigma_{xx}, \sigma_{zz})$ and

$$\nabla_{(\sigma_{xx}, \sigma_{zz})} F|_{h,u} \cdot S = D$$

since $\nabla_{(\sigma_{xx}h^{2(1-\zeta)},\sigma_{zz}h^{2(\zeta-1)})}F\cdot(h^{2(\zeta-1)}S_{h\sigma_{xx}},h^{2(1-\zeta)}S_{h\sigma_{zz}})=D$ by design.

2.2 Suliciu relaxation of the Riemann problem without source

For all time ranges $t\in[t^n,t^{n+1})$, $n=0\dots N-1$, let us now define at each interface $x_{i+\frac{1}{2}}$, $i\in\mathbb{Z}$, between cells i and $i+1$ the numerical flux functions F_l and F_r

$$\begin{aligned} F_l(q_l,q_r) &= F_0(q_l)-\int_{-\infty}^{0}\Big(R(\xi,q_l,q_r)-q_l\Big)d\xi,\\ F_r(q_l,q_r) &= F_0(q_r)+\int_{0}^{\infty}\Big(R(\xi,q_l,q_r)-q_r\Big)d\xi. \end{aligned} \tag{17}$$

invoking an approximate solution $R\left((x-x_{i+1/2})/(t-t^n),q_i^n,q_{i+1}^n\right)$ to the Riemann problem for (10) with initial condition $q_i^n 1_{x<0}+1_{x>0}q_{i+1}^n$ at $t=t^n$, and *any* F_0.

In this work, we propose as approximate solution that given by Suliciu relaxation

$$R(\xi,q_l,q_r)=L\mathscr{R}(\xi,\mathscr{Q}_l,\mathscr{Q}_r) \tag{18}$$

i.e. the projection (operator L) onto $q\equiv(h,hu,h\sigma_{xx},h\sigma_{zz})$ of the *exact* solution $\mathscr{R}(\xi,\mathscr{Q}_l,\mathscr{Q}_r)$ of the Riemann problem for the system with relaxed pressure

$$\left\{\begin{aligned} \partial_t h+\partial_x(hu)&=0\\ \partial_t(hu)+\partial_x(hu^2+\pi)&=0\\ \partial_t(\sigma_{xx}h^{2(1-\zeta)})+u\partial_x(\sigma_{xx}h^{2(1-\zeta)})&=0\\ \partial_t(\sigma_{zz}h^{2(\zeta-1)})+u\partial_x(\sigma_{zz}h^{2(\zeta-1)})&=0\\ \partial_t(h\pi)+\partial_x(hu\pi+uc^2)&=0\\ \partial_t\left(h(u^2/2+\hat{e})\right)+\partial_x\left(hu(u^2/2+\hat{e})+u\pi\right)&=0\\ \partial_t c+u\partial_x c&=0 \end{aligned}\right. \tag{19}$$

and initial condition given by $(o=l,r)$

$$\mathscr{Q}_o=\Big(h_o,(hu)_o,h_o^{1-2\zeta}(h\sigma_{xx})_o,h_o^{2\zeta-3}(h\sigma_{zz})_o,h_oP(q_o),(hu)_o^2/2h_o+e(q_o),c_o\Big) \tag{20}$$

where $c_o(q_l,q_r)$ are chosen so as to ensure stability, that is the dissipation rule (14) here (see below). Note that (19) is a hyperbolic system which fully decomposes into linearly degenerate eigenfields, so $\mathscr{R}$ has an analytic expression (see formulas in [4, 5]). Note also: the Riemann solver R is consistent under the CFL condition

$$\Delta t^n\le\frac{1}{2}\inf_{i\in\mathbb{Z}}\frac{1}{\Delta x_i}\max\left(u_i^n-c_l(q_i^n,q_{i+1}^n)/h_i^n,u_i^n+c_r(q_i^n,q_{i+1}^n)/h_{i+1}^n\right). \tag{21}$$

It remains to specify a choice of functions c_l,c_r preserving $\mathscr{U}^\ell$ and ensuring (14).

Although it is not clear whether our construction allows one to approximate solutions on any time ranges $t \in [0,T)$, since the series $\sum_n \Delta t^n$ may be bounded uniformly for all space-grid choice ($\sup_i |u_i^n|$ may grow unboundedly as $n \to \infty$), specifying such c_l, c_r fully defines a computable scheme. In particular, (15) then implies that (12) at step (ii) always has at least one solution $q_i^{n+1} \in \mathscr{U}^\ell$ for any Δt^n fixed at step (i). (This can be shown using Brouwer fixed-point theorem like in [1].)

Note however a difficulty here for FENE-P fluids with c_l, c_r. Suliciu relaxation approach (19) was retained at step (i) because the solver often allows one to preserve invariant domains like $\mathscr{U}^\ell$ *and* a dissipation rule (14) through well-chosen c_l, c_r, see e.g. [3, 4, 5]. Indeed, on noting the exact Riemann solution to (19), to get (14) on choosing $G(q_l,q_r) = u\Big(h\big(\frac{u^2}{2}+\hat{e}\big)+\pi\Big)|_{\mathscr{R}(0,q_l,q_r)}$, it is enough that $\forall q_l, q_r \in \mathscr{U}^\ell$

$$q_\xi := L\mathscr{R}(\xi,\mathscr{Q}_l,\mathscr{Q}_r) \in \mathscr{U}^\ell \text{ and } h_\xi^2 \partial_h|_{h^{2-2\zeta}\sigma_{xx},h^{2\zeta-2}\sigma_{zz}} P(q_\xi) \le c_\xi^2 \,,\ \forall \xi \in \mathbb{R} \quad (22)$$

using $c_\xi = c_l(q_l,q_r)$ if $\xi <= u^*$ and $c_\xi = c_r(q_l,q_r)$ if $\xi > u^*$ with $u^* := \frac{c_l u_l + \pi_l + c_r u_r - \pi_r}{c_l + c_r}$.

One can easily propose c_l, c_r satisfying the first condition in (22), i.e.

$$\frac{1}{h_l^*} = \frac{1}{h_l}\left(1 + \frac{c_r(u_r-u_l)+\pi_l-\pi_r}{(c_l/h_l)(c_l+c_r)}\right) > 0 \quad (23)$$

$$\frac{1}{h_r^*} = \frac{1}{h_r}\left(1 + \frac{c_l(u_r-u_l)+\pi_r-\pi_l}{(c_r/h_r)(c_l+c_r)}\right) > 0 \quad (24)$$

as usual for Saint-Venant systems, plus the admissibility conditions ($o = l/r$)

$$(h_o^*)^{2(1-\zeta)}(h_o)^{2(\zeta-1)}\sigma_{zz,o} + (h_o^*)^{2(\zeta-1)}(h_o)^{2(1-\zeta)}\sigma_{xx,o} < \ell \quad (25)$$

for any $\sigma_{zz,o}, \sigma_{xx,o} > 0$ satisfying $\sigma_{zz,o} + \sigma_{xx,o} < \ell$ (FENE-P fluids, see below). But the second condition is usually treated for $\phi_o : h \to h\sqrt{\partial_h|_{h_o^{2-2\zeta}\sigma_{xx,o},h_o^{2\zeta-2}\sigma_{zz,o}} P}$ monotone. Unfortunately, a lengthy (but easy) computation shows that the latter is not monotone here, so the standard method to choose c_l, c_r a priori does not apply.

2.3 Choice of relaxation parameter

Let us treat the first part of (22) as usual and define $c_o = \max(h_o\sqrt{\partial_h P(q_o)} := h_o a_o, \tilde{c}_o)$, $o = l/r$ such that the functions $\tilde{c}_o(q_l,q_r)$ ensure (23–24) and (25).

First, let us inspect (23–24) classically following [7, section3.3]. Denoting $a_l Y_l = (u_l-u_r)_+ + \frac{(\pi_r-\pi_l)_+}{h_l a_l + h_r a_r} \ge 0$, $a_r Y_r = (u_l-u_r)_+ + \frac{(\pi_l-\pi_r)_+}{h_l a_l + h_r a_r} \ge 0$ so $\frac{1}{h_o^*} \ge \frac{1-h_o a_o Y_o/c_o}{h_o}$, it then holds $(h_o^*)^{-1} \ge (h_o)^{-1} y_o > 0$ with $y_o := 1 - \frac{Y_o}{1+\alpha_o Y_o} \in (\frac{\alpha_o - 1}{\alpha_o}, 1]$ provided one chooses $\tilde{c}_o > 0$ such that $c_o \ge h_o a_o(1+\alpha_o Y_o)$ for $\alpha_o > 1$, which yields $h_o^* \in (0, h_o/y_o]$ thus (23–24) in particular.

On the other hand, let us now inspect (25), which rewrites with $h_o^* > 0$

$$w_oA_o + w_o^{-1}B_o < 1 \Leftrightarrow 2A_ow_o \in \left(1-\sqrt{1-4A_oB_o}, 1+\sqrt{1-4A_oB_o}\right) \subset \mathbb{R}_{>0} \quad (26)$$

with $w_o = (h_o^*/h_o)^{2(1-\zeta)}$, $A_o = \sigma_{zz,o}/\ell$, $B_o = \sigma_{xx,o}/\ell$ positive such that $A_o + B_o < 1$ (hence $A_oB_o \le A_o(1-A_o) \le \frac{1}{4}$) and $2(1-\zeta) \in [1,2]$. The upper-bound in (26) is satisfied with $\alpha_o = (w_o^+)^{\frac{1}{2(1-\zeta)}}/((w_o^+)^{\frac{1}{2(1-\zeta)}} - 1) > 1$, on noting

$$(w_o^+)^{\frac{1}{2(1-\zeta)}} := \left((1+\sqrt{1-4A_oB_o})/(2A_o)\right)^{\frac{1}{2(1-\zeta)}} \ge \frac{\alpha_o}{\alpha_o - 1} \ge 1/y_o \ge h_o^*/h_o\,. \quad (27)$$

It remains to ensure the lower bound in (26). Obviously, $w_o^- := \frac{1-\sqrt{1-4A_oB_o}}{2A_o} < 1$ so one only needs to inspect the case $h_o^* \le h_o$. Now, with $a_lW_l = (u_r - u_l)_+ + \frac{(\pi_l - \pi_r)_+}{h_la_l + h_ra_r} \ge 0$, $a_rW_r = (u_r - u_l)_+ + \frac{(\pi_r - \pi_l)_+}{h_la_l + h_ra_r} \ge 0$, if $c_o \ge h_oa_oW_o((w_o^-)^{-\frac{1}{2(1-\zeta)}} - 1)^{-1}$ then holds

$$(w_o^-)^{\frac{1}{2(1-\zeta)}} \le (1 + a_oh_oW_o/c_o)^{-1} \le h_o^*/h_o\,.$$

In the end, we claim the following choices

$$c_l = h_l \max\left(a_l + \alpha_l\left((u_l - u_r)_+ + \frac{(\pi_r - \pi_l)_+}{h_la_l + h_ra_r}\right), \beta_l\left((u_r - u_l)_+ + \frac{(\pi_l - \pi_r)_+}{h_la_l + h_ra_r}\right)\right) \quad (28)$$

$$c_r = h_r \max\left(a_r + \alpha_r\left((u_l - u_r)_+ + \frac{(\pi_l - \pi_r)_+}{h_la_l + h_ra_r}\right), \beta_r\left((u_r - u_l)_+ + \frac{(\pi_r - \pi_l)_+}{h_la_l + h_ra_r}\right)\right) \quad (29)$$

satisfy simultaneously (23–24) and (25) in a compatible way with $a_o = \sqrt{\partial_hP(q_o)}$, $\alpha_o = \max(2, (w_o^+)^{\frac{1}{2(1-\zeta)}}/((w_o^+)^{\frac{1}{2(1-\zeta)}} - 1))$, $\beta_o = (w_o^-)^{\frac{1}{2(1-\zeta)}}/(1 - (w_o^-)^{\frac{1}{2(1-\zeta)}}))$, $w_o^- = \frac{\ell - \sqrt{\ell - 4\sigma_{zz,o}\sigma_{xx,o}}}{2\sigma_{zz,o}}$, $w_o^+ = \frac{\ell + \sqrt{\ell - 4\sigma_{zz,o}\sigma_{xx,o}}}{2\sigma_{zz,o}}$, for $o = l/r$. Moreover, note that we have chosen α_o such that all subcharacteristic conditions (22) are satisfied in the $\ell \to \infty$ limit, hence also the free-energy dissipation (15). Indeed, ϕ_o is monotone in the $\ell \to \infty$ limit and one can then apply the standard method to choose c_l, c_r [5].

3 Numerical illustrations

We numerically approximate on $t \in [0, .1]$ the solution to a Riemann problem with

$$\begin{cases} (h_l, u_l, \sigma_{xx,l}, \sigma_{zz,l}) = (1, 0, 1, 1) & x < .5 \\ (h_r, u_r, \sigma_{xx,r}, \sigma_{zz,r}) = (.1, 0, 1, 1) & x > .5 \end{cases}$$

as initial condition when $g = 10$, $\zeta = 0$, $G = .1$, $\lambda = .1$. In Fig. 1, we show the initial condition and the result at $t = .1$ when $\Delta x = 2^{-8}$ for $\ell = 10, 100, 1000$. Note the influence of the parameter ℓ on the stretch $\sigma_{xx} + \sigma_{zz}$. On computing numerically

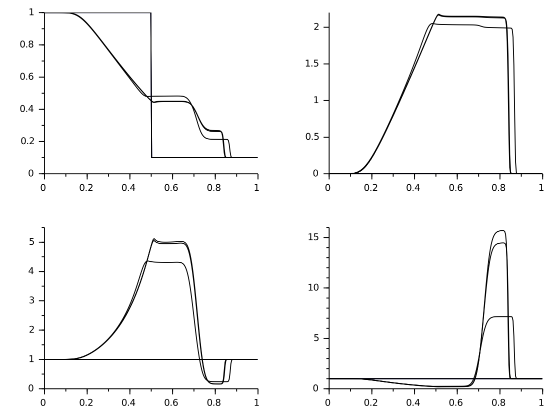

Fig. 1 Top: h (left) and u (right), bottom: σ_{xx} and σ_{zz}.

the free-energy dissipation with the choice of relaxation parameter above, we have never observed the wrong sign, while the time-step did not go to zero.

References

1. Barrett, J.W., Boyaval, S.: Existence and approximation of a (regularized) Oldroyd-B model. M3AS **21**(9), 1783–1837 (2011).
2. Berthon, C., Coquel, F., LeFloch, P.G.: Why many theories of shock waves are necessary: kinetic relations for non-conservative systems. Proceedings of the Royal Society of Edinburgh: Section A Mathematics **142**, 1–37 (2012)
3. Bouchut, F.: Entropy satisfying flux vector splittings and kinetic BGK models. Numerische Mathematik **94**, 623–672 (2003).
4. Bouchut, F.: Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2004)
5. Bouchut, F., Boyaval, S.: A new model for shallow viscoelastic fluids. M3AS **23**(08), 1479–1526 (2013).
6. Bouchut, F., Boyaval, S.: Unified derivation of thin-layer reduced models for shallow free-surface gravity flows of viscous fluids. European Journal of Mechanics - B/Fluids **55, Part 1**, 116–131 (2016).
7. Bouchut, F., Klingenberg, C., Waagan, K.: A multiwave approximate Riemann solver for ideal MHD based on relaxation II: numerical implementation with 3 and 5 waves. Numer. Math. **115**(4), 647–679 (2010).
8. Boyaval, S.: Johnson-Segalman – Saint-Venant equations for viscoelastic shallow flows in the elastic limit. ArXiv e-prints (2016)
9. Dafermos, C.M.: Hyperbolic conservation laws in continuum physics, vol. GM 325. Springer-Verlag, Berlin (2000)
10. Harten, A., Lax, P.D., van Leer, B.: On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. SIAM Rev. **25**(1), 35–61 (1983).
11. LeFloch, P.G.: Hyperbolic systems of conservation laws. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2002). The theory of classical and nonclassical shock waves