

# A Finite-Volume discretization of viscoelastic Saint-Venant equations for FENE-P fluids

Sébastien Boyaval

► **To cite this version:**

Sébastien Boyaval. A Finite-Volume discretization of viscoelastic Saint-Venant equations for FENE-P fluids. Clément Cancès; Pascal Omnes. Finite Volumes for Complex Applications VIII - Hyperbolic, Elliptic and Parabolic Problems. FVCA 2017. Springer Proceedings in Mathematics & Statistics, 200, Springer, pp.163-170, 2017, Print ISBN : 978-3-319-57393-9 / online : 978-3-319-57394-6. <10.1007/978-3-319-57394-6\_18>. <[https://rd.springer.com/chapter/10.1007/978-3-319-57394-6\\_18](https://rd.springer.com/chapter/10.1007/978-3-319-57394-6_18)>. <hal-01433712>

**HAL Id: hal-01433712**

**<https://hal-enpc.archives-ouvertes.fr/hal-01433712>**

Submitted on 12 Jan 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Finite-Volume discretization of viscoelastic Saint-Venant equations for FENE-P fluids

Sébastien Boyaval

**Abstract** Saint-Venant equations can be generalized to account for a viscoelastic rheology in shallow flows. A Finite-Volume discretization for the 1D Saint-Venant system generalized to Upper-Convected Maxwell (UCM) fluids was proposed in [Bouchut & Boyaval, 2013], which preserved a physically-natural stability property (i.e. free-energy dissipation) of the full system. It invoked a relaxation scheme of Suliciu type for the numerical computation of approximate solution to Riemann problems. Here, the approach is extended to the 1D Saint-Venant system generalized to the finitely-extensible nonlinear elastic fluids of Peterlin (FENE-P). We are currently not able to ensure all stability conditions a priori, but numerical simulations went smoothly in a practically useful range of parameters.

**Key words:** Saint-Venant equations, FENE-P viscoelastic fluids, Finite-Volume, simple Riemann solver, Suliciu relaxation scheme

**MSC (2010):** 65M08, 65N08, 35Q30

## 1 Introduction

Saint-Venant equations standardly model shallow free-surface gravity flows and can be generalized to account for the viscoelastic rheology of non-Newtonian fluids [6], Upper-Convected Maxwell (UCM) fluids in particular [5]. Here, we consider a generalized Saint-Venant (gSV) system for *finitely-extensible nonlinear elastic* fluids with Peterlin closure (FENE-P fluids) in Cartesian coordinates

---

Sébastien Boyaval  
Laboratoire d'hydraulique Saint-Venant (Ecole des Ponts ParisTech – EDF R& D – CEREMA)  
Université Paris-Est, EDF'lab 6 quai Watier 78401 Chatou Cedex France, & INRIA Paris MATH-  
ERIALS e-mail: sebastien.boyaval@enpc.fr

$$\partial_t h + \partial_x(hu) = 0 \quad (1)$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2 + hN) = 0 \quad (2)$$

$$\lambda(\partial_t \sigma_{xx} + u \partial_x \sigma_{xx} + 2(\zeta - 1)\sigma_{xx} \partial_x u) = 1 - \sigma_{xx}/(1 - (\sigma_{zz} + \sigma_{xx})/\ell) \quad (3)$$

$$\lambda(\partial_t \sigma_{zz} + u \partial_x \sigma_{zz} + 2(1 - \zeta)\sigma_{zz} \partial_x u) = 1 - \sigma_{zz}/(1 - (\sigma_{zz} + \sigma_{xx})/\ell) \quad (4)$$

for 1D  $\mathbf{e}_y$ -translation invariant flow along  $\mathbf{e}_x$  under a uniform gravity field  $-\mathbf{g}\mathbf{e}_z$  with

- mean flow depth  $h(t, x) > 0$  (in case of a non-rugous flat bottom),
- mean flow velocity  $u(t, x)$  (for *uniform* cross sections), and
- a normal-stress difference  $N = G(\sigma_{zz} - \sigma_{xx})/(1 - (\sigma_{zz} + \sigma_{xx})/\ell)$  given by conformation variables  $\sigma_{zz}, \sigma_{xx} > 0$  constrained by  $0 < \sigma_{zz} + \sigma_{xx} < \ell$ , a relaxation time  $\lambda \geq 0$  and an elasticity modulus  $G > 0$ .

Note that (1-2-3-4) formally reduces to the standard viscous Saint-Venant system with viscosity  $\nu \equiv 2\lambda G \geq 0$  when  $\ell \rightarrow \infty$ ,  $\lambda \rightarrow 0$  and  $G\lambda < \infty$ . Moreover we have used the quite general Gordon-Schwalter derivatives with slip parameter  $\zeta \in [0, \frac{1}{2}]$  constrained by the hyperbolicity of the system (1-2-3-4). (This follows after an easy computation similar to [8].)

In this work, we discuss a Finite-Volume method to solve (numerically) the Cauchy problem for the nonlinear hyperbolic 1D system (1-2-3-4). Standardly, we need to consider *weak* solutions (in fact, to (6-7-8-9), see below) plus *admissibility* constraints that are physically-meaningful dissipation rules formalizing the thermodynamics second principle close to an equilibrium [9]. Here, we consider the *inequality* associated with the companion conservation law for the *free-energy*

$$F = h \left( \frac{u^2}{2} + \frac{gh}{2} - \frac{G}{2(1-\zeta)} (\ell \log((\ell - (\sigma_{xx} + \sigma_{zz}))/(\ell - 2)) + \log(\sigma_{xx}\sigma_{zz})) \right)$$

that is, on denoting the impulse by  $P = gh^2/2 + hN$ ,

$$\begin{aligned} -\frac{Gh}{2(1-\zeta)\lambda} \left( \sigma_{xx}^{-1} \left( 1 - \frac{\sigma_{xx}}{1 - (\sigma_{zz} + \sigma_{xx})/\ell} \right)^2 + \sigma_{zz}^{-1} \left( 1 - \frac{\sigma_{zz}}{1 - (\sigma_{zz} + \sigma_{xx})/\ell} \right)^2 \right) \\ =: D \geq \partial_t F + \partial_x(u(F + P)) \quad (5) \end{aligned}$$

where the left-hand-side is obviously non-positive on the admissibility domain

$$\mathcal{U}^\ell := \{0 < h, 0 < \sigma_{xx}, 0 < \sigma_{zz}, \sigma_{xx} + \sigma_{zz} < \ell\}.$$

Note that we do not consider the vacuum state  $h = 0$  as admissible here, see [8].

## 2 Finite-Volume discretization of FENE-P/Saint-Venant

Piecewise-constant approximate solutions to the Cauchy problem on  $(t, x) \in [0, T] \times \mathbb{R}$  for the gSV system can be defined by a Finite-Volume (FV) method. With a view to preserving  $\mathcal{U}^\ell$  and the dissipation (5) after discretization by a FV method, we choose  $q = (h, hu, h\sigma_{xx}, h\sigma_{zz})$  as discretization variable. Indeed, the free-energy functional  $F$  is *convex* on the convex domain  $\mathcal{U}^\ell \ni q$  (this follows after an easy computation from [4, Lemma 1.3]) while it is not convex in the variable  $(h, hu, h\Pi, h\Sigma)$  whatever smooth invertible functions  $\varpi, \zeta$  are used for the reformulation of gSV

$$\partial_t h + \partial_x(hu) = 0 \quad (6)$$

$$\partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2} + hN\right) = 0 \quad (7)$$

$$\partial_t(h\Pi) + \partial_x(hu\Pi) = \frac{h^{3-2\zeta}\varpi'(\sigma_{xx}h^{2(1-\zeta)})}{\lambda} \left(1 - \frac{\sigma_{xx}}{1 - \frac{\sigma_{zz} + \sigma_{xx}}{\ell}}\right) \quad (8)$$

$$\partial_t(h\Sigma) + \partial_x(hu\Sigma) = \frac{h^{2\zeta-1}\zeta'(\sigma_{zz}h^{2(\zeta-1)})}{\lambda} \left(1 - \frac{\sigma_{zz}}{1 - \frac{\sigma_{zz} + \sigma_{xx}}{\ell}}\right) \quad (9)$$

with  $\Pi = \varpi(\sigma_{xx}h^{2(1-\zeta)})$ ,  $\Sigma = \zeta(\sigma_{zz}h^{2(\zeta-1)})$  (computations are similar to [5, Appendix]). In the sequel, we therefore discretize a quasilinear system with source

$$\partial_t q + A(q)\partial_x q = S(q), \quad (10)$$

which we recall is not ambiguous here (for those discontinuous solutions built using a Riemann solver, at least) thanks to the dissipation rule (5), see [11, 2, 8].

### 2.1 Splitting-in-time

In cell  $(x_{i-1/2}, x_{i+1/2})$ ,  $i \in \mathbb{Z}$ , with volume  $\Delta x_i = x_{i+1/2} - x_{i-1/2} > 0$  and center  $x_i = (x_{i-1/2} + x_{i+1/2})/2$ , we approximate  $q$  solution to (10) on  $\mathbb{R}_{\geq 0} \times \mathbb{R} \ni (t, x)$  by

$$q_i^{n+1} \approx \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} q(t, x) dx, \quad i \in \mathbb{Z}, t \in (t^n, t^{n+1}]$$

on a time grid  $0 = t^0 < t^1 < \dots < t^n < t^{n+1} < \dots < t^N = T$  where  $\Delta t^n = |t^{n+1} - t^n|$  will be chosen small enough compared with  $\Delta x = \sup_{i \in \mathbb{Z}} \Delta x_i < \infty$  to ensure stability.

More precisely, having in mind the numerical approximation of a (well-posed) Cauchy problem for (10) on  $\mathbb{R}_{\geq 0} \times \mathbb{R}$  with initial condition  $q(t \rightarrow 0^+) = q^0 \in L^\infty(\mathbb{R})$ , and therefore starting from approximations  $q_i^0 \approx \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} q^0(x) dx$ ,  $i \in \mathbb{Z}$ , we shall define the cell values  $q_i^n$  in two steps for each  $n = 1, \dots, N$ :

(i) an approximate solution to the *homogeneous* gSV system (i.e. without the source

term  $S$ ) on  $[t^n, t^{n+1})$  is first computed by an explicit three-point scheme

$$q_i^{n+1/2} = q_i^n - \frac{\Delta t^n}{\Delta x_i} (F_l(q_i^n, q_{i+1}^n) - F_r(q_{i-1}^n, q_i^n)), \quad (11)$$

(ii) an approximate solution to the full gSV system on  $(t^n, t^{n+1}]$  is next computed as

$$q_i^{n+1} = q_i^{n+1/2} + \Delta t^n S(q_i^{n+1}). \quad (12)$$

Then, the scheme is consistent with weak solutions of (1–2) equiv. (6–7)

$$q_i^{n+1} = q_i^n - \frac{\Delta t^n}{\Delta x_i} (F_l(q_i^n, q_{i+1}^n) - F_r(q_{i-1}^n, q_i^n)) + \Delta t^n S(q_i^{n+1}) \quad (13)$$

provided the two first flux components for the conservative part  $(h, hu)$  of the variable  $q$  (actually solutions to conservation laws) are conservative  $F_{l,h} = F_{r,h} := F_h$ ,  $F_{l,hu} = F_{r,hu} := F_{hu}$  and consistent  $F_h(q, q) = hu|_q$ ,  $F_{hu}(q, q) = (hu^2 + gh^2/2 + hN)|_q$  as usual, and with the conservative interpretation (8–9) of (3–4) insofar as we next define  $F_l$  and  $F_r$  using a *simple* approximate Riemann solver [10] for (6–7–8–9).

Moreover, with a view to preserving  $\mathcal{U}^\ell$  and a discrete version of (5)

$$F(q_i^{n+1/2}) - F(q_i^n) + \frac{\Delta t^n}{\Delta x_i} (G(q_i^n, q_{i+1}^n) - G(q_{i-1}^n, q_i^n)) \leq 0 \quad (14)$$

for a numerical free-energy flux function consistent with  $G(q, q) = u(F + P)|_q$  in (5), in the sequel, we shall discuss the relaxation technique introduced by Suliciu as simple Riemann solver in step (i), because it proved satisfying for other close systems [3, 4, 5] equipped with an “entropy” convex in the discretization variable like  $F$  here. In the end, for the full scheme (13), a consistent free-energy dissipation

$$F(q_i^{n+1}) - F(q_i^n) + \frac{\Delta t^n}{\Delta x_i} (G(q_i^n, q_{i+1}^n) - G(q_{i-1}^n, q_i^n)) \leq \Delta t^n D(q_i^{n+1}) \quad (15)$$

then holds insofar  $h_i^{n+1/2} = h_i^{n+1}$ ,  $u_i^{n+1/2} = u_i^{n+1}$  and the convexity of  $F$  imply

$$F(q_i^{n+1}) - F(q_i^{n+1/2}) \leq \Delta t^n D(q_i^{n+1}) \leq 0. \quad (16)$$

*Proof.* On noting  $h_i^{n+1/2} = h_i^{n+1}$ ,  $u_i^{n+1/2} = u_i^{n+1}$  it suffices to show that

$$\begin{aligned} \lambda \left( \sigma_{xx,i}^{n+1} - \sigma_{xx,i}^n \right) / \Delta t^n &= 1 - \sigma_{xx,i}^{n+1} / (1 - (\sigma_{zz,i}^{n+1} + \sigma_{xx,i}^{n+1}) / \ell) \\ \lambda \left( \sigma_{zz,i}^{n+1} - \sigma_{zz,i}^n \right) / \Delta t^n &= 1 - \sigma_{zz,i}^{n+1} / (1 - (\sigma_{zz,i}^{n+1} + \sigma_{xx,i}^{n+1}) / \ell) \end{aligned}$$

imply (16). Now, this is obvious, on noting the convexity of  $F|_{h,u}$  in  $(\sigma_{xx}, \sigma_{zz})$  and

$$\nabla_{(\sigma_{xx}, \sigma_{zz})} F|_{h,u} \cdot S = D$$

since  $\nabla_{(\sigma_{xx}h^{2(1-\zeta)}, \sigma_{zz}h^{2(\zeta-1)})} F \cdot (h^{2(\zeta-1)}S_{h\sigma_{xx}}, h^{2(1-\zeta)}S_{h\sigma_{zz}}) = D$  by design.

## 2.2 Suliciu relaxation of the Riemann problem without source

For all time ranges  $t \in [t^n, t^{n+1})$ ,  $n = 0 \dots N-1$ , let us now define at each interface  $x_{i+\frac{1}{2}}$ ,  $i \in \mathbb{Z}$ , between cells  $i$  and  $i+1$  the numerical flux functions  $F_l$  and  $F_r$

$$\begin{aligned} F_l(q_l, q_r) &= F_0(q_l) - \int_{-\infty}^0 \left( R(\xi, q_l, q_r) - q_l \right) d\xi, \\ F_r(q_l, q_r) &= F_0(q_r) + \int_0^{\infty} \left( R(\xi, q_l, q_r) - q_r \right) d\xi. \end{aligned} \quad (17)$$

invoking an approximate solution  $R((x - x_{i+1/2})/(t - t^n), q_i^n, q_{i+1}^n)$  to the Riemann problem for (10) with initial condition  $q_i^n 1_{x < 0} + 1_{x > 0} q_{i+1}^n$  at  $t = t^n$ , and any  $F_0$ .

In this work, we propose as approximate solution that given by Suliciu relaxation

$$R(\xi, q_l, q_r) = L\mathcal{R}(\xi, \mathcal{Q}_l, \mathcal{Q}_r) \quad (18)$$

i.e. the projection (operator  $L$ ) onto  $q \equiv (h, hu, h\sigma_{xx}, h\sigma_{zz})$  of the *exact* solution  $\mathcal{R}(\xi, \mathcal{Q}_l, \mathcal{Q}_r)$  of the Riemann problem for the system with relaxed pressure

$$\left\{ \begin{array}{l} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2 + \pi) = 0 \\ \partial_t(\sigma_{xx}h^{2(1-\zeta)}) + u\partial_x(\sigma_{xx}h^{2(1-\zeta)}) = 0 \\ \partial_t(\sigma_{zz}h^{2(\zeta-1)}) + u\partial_x(\sigma_{zz}h^{2(\zeta-1)}) = 0 \\ \partial_t(h\pi) + \partial_x(hu\pi + uc^2) = 0 \\ \partial_t(h(u^2/2 + \hat{e})) + \partial_x(hu(u^2/2 + \hat{e}) + u\pi) = 0 \\ \partial_t c + u\partial_x c = 0 \end{array} \right. \quad (19)$$

and initial condition given by ( $o = l, r$ )

$$\mathcal{Q}_o = \left( h_o, (hu)_o, h_o^{1-2\zeta}(h\sigma_{xx})_o, h_o^{2\zeta-3}(h\sigma_{zz})_o, h_o P(q_o), (hu)_o^2/2h_o + e(q_o), c_o \right) \quad (20)$$

where  $c_o(q_l, q_r)$  are chosen so as to ensure stability, that is the dissipation rule (14) here (see below). Note that (19) is a hyperbolic system which fully decomposes into linearly degenerate eigenfields, so  $\mathcal{R}$  has an analytic expression (see formulas in [4, 5]). Note also: the Riemann solver  $R$  is consistent under the CFL condition

$$\Delta t^n \leq \frac{1}{2} \inf_{i \in \mathbb{Z}} \frac{1}{\Delta x_i} \max \left( u_i^n - c_l(q_i^n, q_{i+1}^n)/h_i^n, u_i^n + c_r(q_i^n, q_{i+1}^n)/h_{i+1}^n \right). \quad (21)$$

It remains to specify a choice of functions  $c_l, c_r$  preserving  $\mathcal{U}^\ell$  and ensuring (14).

Although it is not clear whether our construction allows one to approximate solutions on any time ranges  $t \in [0, T)$ , since the series  $\sum_n \Delta t^n$  may be bounded uniformly for all space-grid choice ( $\sup_i |u_i^n|$  may grow unboundedly as  $n \rightarrow \infty$ ), specifying such  $c_l, c_r$  fully defines a computable scheme. In particular, (15) then implies that (12) at step (ii) always has at least one solution  $q_i^{n+1} \in \mathcal{U}^\ell$  for any  $\Delta t^n$  fixed at step (i). (This can be shown using Brouwer fixed-point theorem like in [1].)

Note however a difficulty here for FENE-P fluids with  $c_l, c_r$ . Suliciu relaxation approach (19) was retained at step (i) because the solver often allows one to preserve invariant domains like  $\mathcal{U}^\ell$  and a dissipation rule (14) through well-chosen  $c_l, c_r$ , see e.g. [3, 4, 5]. Indeed, on noting the exact Riemann solution to (19), to get (14) on choosing  $G(q_l, q_r) = u \left( h \left( \frac{u^2}{2} + \hat{\varepsilon} \right) + \pi \right) |_{\mathcal{R}(0, q_l, q_r)}$ , it is enough that  $\forall q_l, q_r \in \mathcal{U}^\ell$

$$q_\xi := L\mathcal{R}(\xi, \mathcal{Q}_l, \mathcal{Q}_r) \in \mathcal{U}^\ell \text{ and } h_\xi^2 \partial_h |_{h^{2-2\zeta} \sigma_{xx}, h^{2\zeta-2} \sigma_{zz}} P(q_\xi) \leq c_\xi^2, \forall \xi \in \mathbb{R} \quad (22)$$

using  $c_\xi = c_l(q_l, q_r)$  if  $\xi \leq u^*$  and  $c_\xi = c_r(q_l, q_r)$  if  $\xi > u^*$  with  $u^* := \frac{c_l u_l + \pi_l + c_r u_r - \pi_r}{c_l + c_r}$ .

One can easily propose  $c_l, c_r$  satisfying the first condition in (22), i.e.

$$\frac{1}{h_l^*} = \frac{1}{h_l} \left( 1 + \frac{c_r(u_r - u_l) + \pi_l - \pi_r}{(c_l/h_l)(c_l + c_r)} \right) > 0 \quad (23)$$

$$\frac{1}{h_r^*} = \frac{1}{h_r} \left( 1 + \frac{c_l(u_r - u_l) + \pi_r - \pi_l}{(c_r/h_r)(c_l + c_r)} \right) > 0 \quad (24)$$

as usual for Saint-Venant systems, plus the admissibility conditions ( $o = l/r$ )

$$(h_o^*)^{2(1-\zeta)} (h_o)^{2(\zeta-1)} \sigma_{zz,o} + (h_o^*)^{2(\zeta-1)} (h_o)^{2(1-\zeta)} \sigma_{xx,o} < \ell \quad (25)$$

for any  $\sigma_{zz,o}, \sigma_{xx,o} > 0$  satisfying  $\sigma_{zz,o} + \sigma_{xx,o} < \ell$  (FENE-P fluids, see below). But the second condition is usually treated for  $\phi_o : h \rightarrow h \sqrt{\partial_h |_{h_o^{2-2\zeta} \sigma_{xx,o}, h_o^{2\zeta-2} \sigma_{zz,o}} P}$  monotone. Unfortunately, a lengthy (but easy) computation shows that the latter is not monotone here, so the standard method to choose  $c_l, c_r$  a priori does not apply.

### 2.3 Choice of relaxation parameter

Let us treat the first part of (22) as usual and define  $c_o = \max(h_o \sqrt{\partial_h P(q_o)}) := h_o a_o, \tilde{c}_o$ ,  $o = l/r$  such that the functions  $\tilde{c}_o(q_l, q_r)$  ensure (23–24) and (25).

First, let us inspect (23–24) classically following [7, section 3.3]. Denoting  $a_l Y_l = (u_l - u_r)_+ + \frac{(\pi_r - \pi_l)_+}{h_l a_l + h_r a_r} \geq 0$ ,  $a_r Y_r = (u_l - u_r)_+ + \frac{(\pi_l - \pi_r)_+}{h_l a_l + h_r a_r} \geq 0$  so  $\frac{1}{h_o^*} \geq \frac{1 - h_o a_o Y_o / c_o}{h_o}$ , it then holds  $(h_o^*)^{-1} \geq (h_o)^{-1} y_o > 0$  with  $y_o := 1 - \frac{Y_o}{1 + \alpha_o Y_o} \in (\frac{\alpha_o - 1}{\alpha_o}, 1]$  provided one chooses  $\tilde{c}_o > 0$  such that  $c_o \geq h_o a_o (1 + \alpha_o Y_o)$  for  $\alpha_o > 1$ , which yields  $h_o^* \in (0, h_o / y_o]$  thus (23–24) in particular.

On the other hand, let us now inspect (25), which rewrites with  $h_o^* > 0$

$$w_o A_o + w_o^{-1} B_o < 1 \Leftrightarrow 2A_o w_o \in \left(1 - \sqrt{1 - 4A_o B_o}, 1 + \sqrt{1 - 4A_o B_o}\right) \subset \mathbb{R}_{>0} \quad (26)$$

with  $w_o = (h_o^*/h_o)^{2(1-\zeta)}$ ,  $A_o = \sigma_{zz,o}/\ell$ ,  $B_o = \sigma_{xx,o}/\ell$  positive such that  $A_o + B_o < 1$  (hence  $A_o B_o \leq A_o(1 - A_o) \leq \frac{1}{4}$ ) and  $2(1 - \zeta) \in [1, 2]$ . The upper-bound in (26) is satisfied with  $\alpha_o = (w_o^+)^{\frac{1}{2(1-\zeta)}} / ((w_o^+)^{\frac{1}{2(1-\zeta)}} - 1) > 1$ , on noting

$$(w_o^+)^{\frac{1}{2(1-\zeta)}} := \left( (1 + \sqrt{1 - 4A_o B_o}) / (2A_o) \right)^{\frac{1}{2(1-\zeta)}} \geq \frac{\alpha_o}{\alpha_o - 1} \geq 1/y_o \geq h_o^*/h_o. \quad (27)$$

It remains to ensure the lower bound in (26). Obviously,  $w_o^- := \frac{1 - \sqrt{1 - 4A_o B_o}}{2A_o} < 1$  so one only needs to inspect the case  $h_o^* \leq h_o$ . Now, with  $a_l W_l = (u_r - u_l)_+ + \frac{(\pi_l - \pi_r)_+}{h_l a_l + h_r a_r} \geq 0$ ,  $a_r W_r = (u_r - u_l)_+ + \frac{(\pi_r - \pi_l)_+}{h_l a_l + h_r a_r} \geq 0$ , if  $c_o \geq h_o a_o W_o ((w_o^-)^{-\frac{1}{2(1-\zeta)}} - 1)^{-1}$  then holds

$$(w_o^-)^{\frac{1}{2(1-\zeta)}} \leq (1 + a_o h_o W_o / c_o)^{-1} \leq h_o^*/h_o.$$

In the end, we claim the following choices

$$\begin{aligned} c_l &= h_l \max \left( a_l + \alpha_l \left( (u_l - u_r)_+ + \frac{(\pi_r - \pi_l)_+}{h_l a_l + h_r a_r} \right), \beta_l \left( (u_r - u_l)_+ + \frac{(\pi_l - \pi_r)_+}{h_l a_l + h_r a_r} \right) \right) \quad (28) \\ c_r &= h_r \max \left( a_r + \alpha_r \left( (u_l - u_r)_+ + \frac{(\pi_l - \pi_r)_+}{h_l a_l + h_r a_r} \right), \beta_r \left( (u_r - u_l)_+ + \frac{(\pi_r - \pi_l)_+}{h_l a_l + h_r a_r} \right) \right) \quad (29) \end{aligned}$$

satisfy simultaneously (23–24) and (25) in a compatible way with  $a_o = \sqrt{\partial_h P(q_o)}$ ,  $\alpha_o = \max(2, (w_o^+)^{\frac{1}{2(1-\zeta)}} / ((w_o^+)^{\frac{1}{2(1-\zeta)}} - 1))$ ,  $\beta_o = (w_o^-)^{\frac{1}{2(1-\zeta)}} / (1 - (w_o^-)^{\frac{1}{2(1-\zeta)}})$ ,  $w_o^- = \frac{\ell - \sqrt{\ell - 4\sigma_{zz,o}\sigma_{xx,o}}}{2\sigma_{zz,o}}$ ,  $w_o^+ = \frac{\ell + \sqrt{\ell - 4\sigma_{zz,o}\sigma_{xx,o}}}{2\sigma_{zz,o}}$ , for  $o = l/r$ . Moreover, note that we have chosen  $\alpha_o$  such that all subcharacteristic conditions (22) are satisfied in the  $\ell \rightarrow \infty$  limit, hence also the free-energy dissipation (15). Indeed,  $\phi_o$  is monotone in the  $\ell \rightarrow \infty$  limit and one can then apply the standard method to choose  $c_l, c_r$  [5].

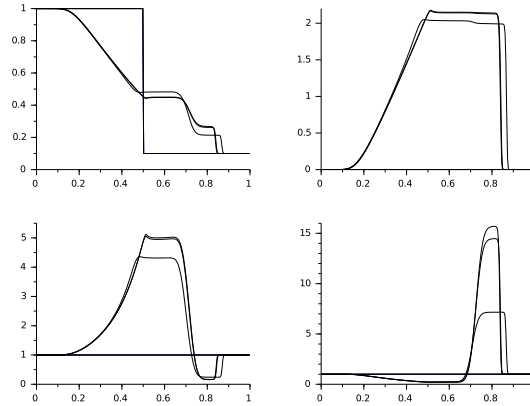
### 3 Numerical illustrations

We numerically approximate on  $t \in [0, .1]$  the solution to a Riemann problem with

$$\begin{cases} (h_l, u_l, \sigma_{xx,l}, \sigma_{zz,l}) = (1, 0, 1, 1) & x < .5 \\ (h_r, u_r, \sigma_{xx,r}, \sigma_{zz,r}) = (.1, 0, 1, 1) & x > .5 \end{cases}$$

as initial condition when  $g = 10$ ,  $\zeta = 0$ ,  $G = .1$ ,  $\lambda = .1$ . In Fig. 1, we show the initial condition and the result at  $t = .1$  when  $\Delta x = 2^{-8}$  for  $\ell = 10, 100, 1000$ . Note the influence of the parameter  $\ell$  on the stretch  $\sigma_{xx} + \sigma_{zz}$ . On computing numerically





**Fig. 1** Top:  $h$  (left) and  $u$  (right), bottom:  $\sigma_{xx}$  and  $\sigma_{zz}$ .

the free-energy dissipation with the choice of relaxation parameter above, we have never observed the wrong sign, while the time-step did not go to zero.

## References

1. Barrett, J.W., Boyaval, S.: Existence and approximation of a (regularized) Oldroyd-B model. *M3AS* **21**(9), 1783–1837 (2011).
2. Berthon, C., Coquel, F., LeFloch, P.G.: Why many theories of shock waves are necessary: kinetic relations for non-conservative systems. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **142**, 1–37 (2012)
3. Bouchut, F.: Entropy satisfying flux vector splittings and kinetic BGK models. *Numerische Mathematik* **94**, 623–672 (2003).
4. Bouchut, F.: Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources. *Frontiers in Mathematics*. Birkhäuser Verlag, Basel (2004)
5. Bouchut, F., Boyaval, S.: A new model for shallow viscoelastic fluids. *M3AS* **23**(08), 1479–1526 (2013).
6. Bouchut, F., Boyaval, S.: Unified derivation of thin-layer reduced models for shallow free-surface gravity flows of viscous fluids. *European Journal of Mechanics - B/Fluids* **55**, Part 1, 116–131 (2016).
7. Bouchut, F., Klingenberg, C., Waagan, K.: A multiwave approximate Riemann solver for ideal MHD based on relaxation II: numerical implementation with 3 and 5 waves. *Numer. Math.* **115**(4), 647–679 (2010).
8. Boyaval, S.: Johnson-Segalman – Saint-Venant equations for viscoelastic shallow flows in the elastic limit. *ArXiv e-prints* (2016)
9. Dafermos, C.M.: *Hyperbolic conservation laws in continuum physics*, vol. GM 325. Springer-Verlag, Berlin (2000)
10. Harten, A., Lax, P.D., van Leer, B.: On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Rev.* **25**(1), 35–61 (1983).
11. LeFloch, P.G.: *Hyperbolic systems of conservation laws. Lectures in Mathematics ETH Zürich*. Birkhäuser Verlag, Basel (2002). The theory of classical and nonclassical shock waves