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# Asymptotic normality and combinatorial aspects of the prefix exchange distance distribution

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## Abstract

The prefix exchange distance of a permutation is the minimum number of exchanges involving the leftmost element that sorts the permutation. We give new combinatorial proofs of known results on the distribution of the prefix exchange distance for a random uniform permutation. We also obtain expressions for the mean and the variance of this distribution, and finally, we show that the normalised prefix exchange distribution converges in distribution to the standard normal distribution.

*Keywords:* star poset, Whitney numbers, combinatorial proofs, permutation, distance, prefix exchange, distribution, asymptotic normality

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## 1. Introduction

An ever-growing body of research has been devoted to the study of various measures of disorder on permutations, with the intention of expressing how many elementary operations (whose type may vary but which are fixed beforehand) they should undergo in order to become sorted. One of the earliest examples of such a measure is the *Cayley distance*, which corresponds to the minimum number of transpositions that must be applied to a permutation in order to obtain the identity permutation. That distance is easily expressed in

terms of the number of *cycles* of the permutation [5], and the *signless Stirling numbers of the first kind* can be used to characterise exactly the distribution of the Cayley distance — i.e. the number of permutations of  $n$  elements with Cayley distance  $k$ . Motivations for studying those distances and their distributions outside pure mathematical fields include the study of sorting algorithms [8], genome comparison [9], and the design of interconnection networks [13].

We focus in this paper on the *prefix exchange* operation, a restricted kind of transposition that swaps any element of a permutation with its first element. This operation was introduced by Akers and Krishnamurthy [1], who also gave a formula for computing the associated *prefix exchange distance*, i.e. the minimum number of prefix exchanges required to transform a given permutation into the identity permutation. Portier and Vaughan [15] later succeeded in obtaining the generating function of the corresponding distribution, which they then used to derive an explicit formula (with subsequent corrections by Shen and Qiu [18]) as well as recurrence formulas for computing the so-called “Whitney numbers of the second kind for the star poset”, i.e. the number of permutations of size  $n$  with prefix exchange distance  $k$  (see Portier [14] for a table with the first few terms).

We revisit in this paper the results obtained by Portier and Vaughan [15] by taking the opposite direction: we first obtain new proofs for their exact and recurrence formulas, and then use those formulas to recover their expression for the generating function. Our proofs are purely combinatorial, a desirable property since such proofs are often simpler in addition to providing new insight into the underlying objects [4, 19]. We then proceed to obtaining the mean and the variance of the distribution, and finally, we examine the behaviour of this distribution when  $n$  tends to infinity: in particular, we show that the normalised prefix exchange distribution converges in distribution to the standard normal distribution.

## 2. Background and known results

We recall some basic notions and notation (see e.g. Bóna [4]) that will be useful throughout the text.

### 2.1. Permutations and cycles

For  $n \geq 1$ , we let  $\mathfrak{S}_n$  denote the *symmetric group*, i.e. the set of all permutations of  $\{1, 2, \dots, n\}$  together with the usual function composition operation  $\circ$  applied from right to left. We view permutations as sequences and denote them using lower case Greek letters, i.e.  $\pi = \langle \pi_1 \ \pi_2 \ \cdots \ \pi_n \rangle$ , where  $\pi_i = \pi(i)$  for  $1 \leq i \leq n$ . We will sometimes find it convenient to *reduce* permutations in the following sense.

**Definition 2.1.** [12] The *reduced form* of a permutation  $\sigma$  of a set  $\{j_1, j_2, \dots, j_r\}$  with  $j_1 < j_2 < \dots < j_r$  is the permutation  $red(\sigma) \in \mathfrak{S}_r$  obtained by replacing  $j_i$  with  $i$  in  $\sigma$  for all  $1 \leq i \leq r$ .

As is well-known, every permutation  $\pi$  decomposes in a single way into disjoint cycles (up to the ordering of cycles and of elements within each cycle). For instance, when  $\pi = \langle 4 \ 1 \ 6 \ 2 \ 5 \ 7 \ 3 \rangle$ , the *disjoint cycle decomposition* is  $\pi = (1, 4, 2)(3, 6, 7)(5)$  (notice the parentheses and the commas). We use  $c_1(\pi)$  to denote the number of cycles of length 1, or *fixed points*, of  $\pi$ , and  $c_{\geq 2}(\pi)$  to denote the number of cycles of length at least 2 of  $\pi$ .

Let  $dcd(\pi)$  denote the disjoint cycle decomposition of  $\pi$ . It will sometimes be convenient to abuse notation by writing, for some permutation  $\pi \in \mathfrak{S}_n$ ,  $\sigma = dcd(\pi) \cup (n+1)$ , to express the fact that the disjoint cycle decomposition of  $\pi$  and  $\sigma$  differ only by the fixed point  $\sigma_{n+1} = n+1$ , which does not exist in  $\pi$ .

Recall that, for  $0 \leq k \leq n$ , the *signless Stirling number of the first kind*  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  counts the number of permutations of  $n$  elements with  $k$  cycles, with the convention that  $\left[ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] = 0$  for  $n > 1$  and  $\left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] = 1$ . Those numbers are well-known to appear in the following series expansion of the *ascending factorial*:

$$x^{\overline{n}} = x(x+1) \cdots (x+n-1) = \sum_{k=0}^n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k. \quad (1)$$

The *signed Stirling number of the first kind* is  $s(n, k) = (-1)^{n-k} \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ .

## 2.2. Prefix exchanges

For every  $i = 2, 3, \dots, n$ , the *prefix exchange*  $(1, i)$  applied to a permutation  $\pi$  in  $\mathfrak{S}_n$  transforms  $\pi$  into  $\pi \circ (1, i)$  by swapping elements  $\pi_1$  and  $\pi_i$ . The *prefix exchange distance* of  $\pi$ , denoted by  $pexc(\pi)$ , is the minimum number of prefix exchanges needed to sort the permutation  $\pi$ , i.e. to transform it into the identity permutation  $\iota = \langle 1 \ 2 \ \dots \ n \rangle$ . Akers et al. [2] prove the following formula for computing the prefix exchange distance:

**Theorem 2.1.** [2] *The prefix exchange distance of  $\pi$  in  $\mathfrak{S}_n$  is equal to*

$$pexc(\pi) = n + c_{\geq 2}(\pi) - c_1(\pi) - \begin{cases} 0 & \text{if } \pi_1 = 1, \\ 2 & \text{otherwise.} \end{cases} \quad (2)$$

Akers et al. [2] refer to the Cayley graph of  $\mathfrak{S}_n$  generated by prefix exchanges as the “ $n$ -star graph”. They show that the diameter of that graph, or equivalently the largest value that the distance can reach, is  $\lfloor 3(n-1)/2 \rfloor$ .

Let  $n \geq 1$  be fixed. For  $k \geq 0$ , we denote  $W_{n,k}$  the number of permutations in  $\mathfrak{S}_n$  which are at prefix exchange distance  $k$  from the identity permutation. These numbers are called in the literature “Whitney numbers of the second kind for the star poset” or “surface areas for the star graph”. An explicit formula for these numbers was first given by Portier and Vaughan [15], and later corrected by Shen and Qiu [18]:

**Theorem 2.2.** [18] *The Whitney numbers of the second kind for the star poset are given as follows. Let  $n \geq 1$  and  $0 \leq k \leq \lfloor 3(n-1)/2 \rfloor$  and denote, for  $0 \leq i \leq \min(n-1, k+1)$ :*

$$T_i = \max \left\{ 0, \left\lceil \frac{k-2i}{2} \right\rceil \right\}, \quad S_i = \min \left\{ n-1-i, \left\lfloor \frac{k+1-i}{2} \right\rfloor \right\}.$$

With these notation, we have:

$$W_{n,k} = \sum_{i=0}^{\min(n-1, k+1)} \sum_{t=T_i}^{S_i} \binom{n-1}{i} \binom{n-1-i}{t} s(i+1, k-i+1-2t) (-1)^{k+2-t}.$$

Using different approaches, Imani et al. [11] and Cheng et al. [7] give alternative explicit formulas for  $W_{n,k}$ . The following recurrence relations are also known (see Portier and Vaughan [15] for the first one and Qiu and Akl [16] for the second and the third):

**Theorem 2.3.** *The Whitney numbers of the second kind for the star poset obey the following recurrence relations: for  $n \geq 1$  and  $3 \leq k \leq \lfloor 3(n-1)/2 \rfloor$ , we have:*

$$W_{n,k} = W_{n-1,k} + (n-1)W_{n-1,k-1} - (n-2)W_{n-2,k-1} + (n-2)W_{n-2,k-3}, \quad (3)$$

$$W_{n,k} = (n-1)W_{n-1,k-1} + \sum_{j=1}^{n-2} jW_{j,k-3}, \quad (4)$$

with  $W_{n,0} = 1$ ,  $W_{n,1} = n-1$  and  $W_{n,2} = (n-1)(n-2)$ . We also have, for  $n \geq 1$  and  $0 \leq k \leq \lfloor \frac{3(n-1)}{2} \rfloor$ :

$$W_{n+k+1,k} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k+1}{i} W_{n+k+1-i,k}. \quad (5)$$

### 3. Combinatorial derivation of the formula for $W_{n,k}$

The explicit formula for  $W_{n,k}$  given in Theorem 2.2 was obtained by Portier and Vaughan [15] (notwithstanding some errors later corrected by Shen and Qiu [18]) using a generating function technique: they first derived the generating function of these numbers and then used it to deduce a formula for  $W_{n,k}$ . We give here a direct combinatorial derivation of the formula in Theorem 2.2 based on *derangements*, i.e. permutations without any fixed point. We proceed in two steps, by first computing the number  $W_{n,k}^{(1)}$  of permutations at distance  $k$  that fix 1 and then the number  $W_{n,k}^{(2)}$  of permutations at distance  $k$  that do not fix 1. We will need the following preliminary result, which counts the number  $d(n,k)$  of derangements in  $\mathfrak{S}_n$  with  $k$  cycles.

**Lemma 3.1.** [17] *For  $1 \leq k \leq n$ , we have*

$$d(n,k) = \sum_{j=0}^k (-1)^j \binom{n}{j} \begin{bmatrix} n-j \\ k-j \end{bmatrix},$$

with the convention  $d(n,0) = 0$ .

The following well-known relation (see e.g. Graham et al. [10, page 167]) will also be useful:

$$\binom{r}{m} \binom{m}{p} = \binom{r}{p} \binom{r-p}{m-p} \quad (6)$$

for any  $m, p, r \in \mathbb{N}$ .

**Proposition 3.1.** *The number of permutations  $\pi$  in  $\mathfrak{S}_n$  with  $\text{pexc}(\pi) = k$  and  $\pi_1 = 1$  is*

$$W_{n,k}^{(1)} = \sum_{\ell=\max(n-k-1,0)}^{\lfloor(2n-k-2)/2\rfloor} \sum_{j=0}^{k-n+\ell+1} \binom{n-1}{\ell+j} \binom{\ell+j}{j} (-1)^j \begin{bmatrix} n-\ell-j-1 \\ k-n+\ell-j+1 \end{bmatrix}. \quad (7)$$

*Proof.* We sum over all possible values  $i$  for the number of fixed points of  $\pi$ . For a permutation  $\pi$  with  $\text{pexc}(\pi) = k$  and  $c_1(\pi) = i$ , Equation (2) implies that  $c_{\geq 2}(\pi) = k - n + i$ . From the conditions  $k - n + i \geq 0$  and  $n - i \geq k - n + i$  we easily obtain the following bounds for  $i$ :  $\max(n - k, 1) \leq i \leq \lfloor(2n - k)/2\rfloor$ .

Since  $\pi_1 = 1$ , there are  $\binom{n-1}{i-1}$  choices for the other  $i - 1$  fixed points. The remaining  $n - i$  elements must form  $k - n + i$  cycles of length at least 2, and there are exactly  $d(n - i, k - n + i)$  ways to do this. We obtain

$$\begin{aligned} W_{n,k}^{(1)} &= \sum_{i=\max(n-k,1)}^{\lfloor(2n-k)/2\rfloor} \binom{n-1}{i-1} d(n-i, k-n+i) \\ &= \sum_{i=\max(n-k,1)}^{\lfloor(2n-k)/2\rfloor} \binom{n-1}{i-1} \sum_{j=0}^{k-n+i} (-1)^j \binom{n-i}{j} \begin{bmatrix} n-i-j \\ k-n+i-j \end{bmatrix} \quad (\text{using Lemma 3.1}). \end{aligned}$$

Setting  $\ell = i - 1$ , we have

$$W_{n,k}^{(1)} = \sum_{\ell=\max(n-k-1,0)}^{\lfloor(2n-k-2)/2\rfloor} \binom{n-1}{\ell} \sum_{j=0}^{k-n+\ell+1} (-1)^j \binom{n-\ell-1}{j} \begin{bmatrix} n-\ell-j-1 \\ k-n+\ell-j+1 \end{bmatrix},$$

and using Equation (6) with  $r = n - 1$ ,  $m = \ell + j$  and  $p = j$  allows us to complete the proof.  $\square$

**Proposition 3.2.** *The number of permutations  $\pi$  in  $\mathfrak{S}_n$  with  $\text{pexc}(\pi) = k$  and  $\pi_1 \neq 1$  is*

$$\begin{aligned} W_{n,k}^{(2)} &= \sum_{i=\max(n-k-2,0)}^{\lfloor(2n-k-2)/2\rfloor} \sum_{j=0}^{k-n+i+2} \binom{n-1}{i+j} \binom{i+j}{j} (-1)^j \begin{bmatrix} n-i-j \\ k-n+i+2-j \end{bmatrix} \\ &+ \sum_{i=\max(n-k-1,0)}^{\lfloor(2n-k-2)/2\rfloor} \sum_{j=1}^{k-n+i+2} \binom{n-1}{i+j-1} \binom{i+j-1}{j-1} (-1)^j \begin{bmatrix} n-i-j \\ k-n+i+2-j \end{bmatrix}. \end{aligned}$$

*Proof.* As in the proof of Proposition 3.1, we sum over all possible values  $i$  for the number of fixed points of  $\pi$ . In this case, if a permutation  $\pi$  has  $i$  fixed points and is at distance  $k$ , then Equation (2) implies that  $c_{\geq 2}(\pi) = k - n + i + 2$ . From the two conditions:  $k - n + i + 2 \geq 0$  and  $n - i \geq k - n + i + 2$ , we derive the following bounds for  $i$ :  $\max(n - k - 2, 0) \leq i \leq \lfloor(2n - k - 2)/2\rfloor$ .

Since  $\pi_1 \neq 1$ , we have  $\binom{n-1}{i}$  choices for the  $i$  fixed points. Furthermore, the remaining  $n-i$  elements must form  $k-n+i+2$  cycles of length at least 2. We obtain:

$$\begin{aligned} W_{n,k}^{(2)} &= \sum_{i=\max(n-k-2,0)}^{\lfloor (2n-k-2)/2 \rfloor} \binom{n-1}{i} d(n-i, k-n+i+2) \\ &= \sum_{i=\max(n-k-2,0)}^{\lfloor (2n-k-2)/2 \rfloor} \binom{n-1}{i} \sum_{j=0}^{k-n+i+2} (-1)^j \binom{n-i}{j} \left[ \begin{matrix} n-i-j \\ k-n+i+2-j \end{matrix} \right] \quad (\text{using Lemma 3.1}). \end{aligned}$$

One can easily check that the following relations hold:

$$\begin{aligned} \binom{n-1}{i} \binom{n-i}{j} &= \binom{n-1}{i+j} \binom{i+j}{j} \frac{n-i}{n-i-j} && (\text{using Equation (6)}) \\ &= \binom{n-1}{i+j} \binom{i+j}{j} \left( 1 + \frac{j}{n-i-j} \right) \\ &= \binom{n-1}{i+j} \binom{i+j}{j} + \binom{n-1}{i+j-1} \binom{i+j-1}{j-1}, \end{aligned}$$

where for the last line we have used the fact that

$$\binom{n-1}{i+j} \binom{i+j}{j} \frac{j}{n-i-j} = \binom{n-1}{i+j-1} \binom{i+j-1}{j-1}.$$

This allows us to obtain the formula in the statement, and the proof is complete.  $\square$

Propositions 3.1 and 3.2 allow us to recover the expression in Theorem 2.2 as follows. First, decompose the expression in Proposition 3.2 into  $S_1$  and  $S_2$ :

$$\begin{aligned} W_{n,k}^{(2)} &= \overbrace{\sum_{i=\max(n-k-2,0)}^{\lfloor (2n-k-2)/2 \rfloor} \sum_{j=0}^{k-n+i+2} \binom{n-1}{i+j} \binom{i+j}{j} (-1)^j \left[ \begin{matrix} n-i-j \\ k-n+i+2-j \end{matrix} \right]}^{S_1} \\ &+ \underbrace{\sum_{i=\max(n-k-1,0)}^{\lfloor (2n-k-2)/2 \rfloor} \sum_{j=1}^{k-n+i+2} \binom{n-1}{i+j-1} \binom{i+j-1}{j-1} (-1)^j \left[ \begin{matrix} n-i-j \\ k-n+i+2-j \end{matrix} \right]}_{S_2}, \end{aligned}$$

and set  $u = j - 1$  in  $S_2$  to obtain

$$S_2 = - \sum_{i=\max(n-k-1,0)}^{\lfloor (2n-k-2)/2 \rfloor} \sum_{u=0}^{k-n+i+1} \binom{n-1}{i+u} \binom{i+u}{u} (-1)^u \left[ \begin{matrix} n-i-u-1 \\ k-n+i-u+1 \end{matrix} \right].$$



Using Equation (7), we note that  $S_2 = -W_{n,k}^{(1)}$ , so  $W_{n,k} = W_{n,k}^{(1)} + W_{n,k}^{(2)} = S_1$ .

If we then set  $\ell = n - i - j - 1$  in  $S_1$ , we obtain

$$W_{n,k} = \sum_{\ell=0}^{\min(n-1,k+1)} \sum_{j=\max(\lceil(k-2\ell)/2\rceil,0)}^{\min(n-1-\ell, \lfloor(k+1-\ell)/2\rfloor)} \binom{n-1}{\ell} \binom{n-1-\ell}{j} (-1)^j \left[ \begin{matrix} \ell+1 \\ k-\ell-2j+1 \end{matrix} \right],$$

and the fact that  $s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$  yields the formula in Theorem 2.2.

#### 4. Combinatorial proof of the recurrence relations

We now turn to the recurrence relations in Theorem 2.3. We will find it convenient to introduce the following additional notation:

$$\begin{aligned} \mathfrak{S}_{n,k} &= \{\pi \in \mathfrak{S}_n \mid \text{perc}(\pi) = k\} \text{ (so } W_{n,k} = |\mathfrak{S}_{n,k}| \text{); and} \\ \mathfrak{S}_{n,k,i} &= \{\pi \in \mathfrak{S}_{n,k} \mid \pi_i = i\}. \end{aligned}$$

##### 4.1. Proof of Equation (3)

Portier and Vaughan [15] prove the recurrence relation in Equation (3) using a generating function technique. We give here a direct combinatorial proof, again distinguishing between permutations that fix the first element and those that do not.

*Proof.* Let  $n \geq 1$  and  $3 \leq k \leq \lfloor 3(n-1)/2 \rfloor$  be fixed.

1. **permutations  $\pi$  in  $\mathfrak{S}_{n,k}$  with  $\pi_1 \neq 1$ :** we compute  $W_{n,k}^{(2)}$  by summing over all permutations  $\pi$  which are at distance  $k$  and verify  $\pi_1 = i$  for a given  $i \in \{2, 3, \dots, n\}$ . For a given  $2 \leq i \leq n$ , we introduce the following mappings:

$$\begin{aligned} \phi_i &: \{\pi \in \mathfrak{S}_{n,k} \mid \pi_1 = i\} \rightarrow \mathfrak{S}_{n,k-1,i} : & \pi &\mapsto \sigma = \pi \circ (1, i), \\ \psi_i &: \mathfrak{S}_{n,k-1,i} \rightarrow \mathfrak{S}_{n-1,k-1} : & \sigma &\mapsto \tau = \text{red}(\text{dcd}(\sigma) \setminus (i)). \end{aligned}$$

Both mappings are bijective and allow us to associate any element  $\pi \in \mathfrak{S}_{n,k}$  with  $\pi_1 = i$  to an element  $\tau = \psi_i(\phi_i(\pi)) \in \mathfrak{S}_{n-1,k-1}$ . Therefore,

$$|\{\pi \in \mathfrak{S}_{n,k} \mid \pi_1 = i\}| = W_{n-1,k-1}. \quad (8)$$

Since this holds for every  $2 \leq i \leq n$ , we obtain

$$W_{n,k}^{(2)} = |\{\pi \in \mathfrak{S}_{n,k} \mid \pi_1 \neq 1\}| = (n-1)W_{n-1,k-1}, \quad (9)$$

which in turn yields

$$W_{n,k} = W_{n,k}^{(1)} + W_{n,k}^{(2)} = W_{n,k}^{(1)} + (n-1)W_{n-1,k-1}. \quad (10)$$

2. **permutations  $\pi$  in  $\mathfrak{S}_{n,k}$  with  $\pi_1 = 1$ :** in order to compute  $W_{n,k}^{(1)}$ , we further distinguish permutations in  $\mathfrak{S}_{n,k,1}$  based on the value of their last element. More precisely, for  $i = 2, 3, \dots, n$ , let  $W_{n,k}^{(1,i)}$  denote the number of permutations  $\pi$  in  $\mathfrak{S}_{n,k,1}$  with  $\pi_n = i$ . We have

$$W_{n,k}^{(1)} = W_{n,k}^{(1,n)} + \sum_{i=2}^{n-1} W_{n,k}^{(1,i)}.$$

We first note that  $W_{n,k}^{(1,n)} = W_{n-1,k}^{(1)}$ , since any permutation  $\pi \in \mathfrak{S}_{n,k,1}$  with  $\pi_n = n$  can be bijectively mapped onto a permutation  $\tau \in \mathfrak{S}_{n-1,k,1}$  by deleting  $\pi_n = n$ .

For  $i \in \{2, 3, \dots, n-1\}$ , we will next compute  $W_{n,k}^{(1,i)}$ . Let  $\pi \in \mathfrak{S}_{n,k,1}$  be a permutation with  $\pi_n = i$ . Then deleting  $\pi_1 = 1$  and renaming element  $n$  into 1 maps  $\pi$  bijectively onto a permutation  $\tau \in \mathfrak{S}_{n-1}$  with  $\tau_1 = i$  and having the same cycle structure as  $\pi$  except for the deleted singleton (1). Using Equation (2), we can easily see that  $\text{perc}(\tau) = k-2$ , so  $\tau \in \mathfrak{S}_{n-1,k-2}$  and Equation (8) implies that the number of such permutations  $\tau$  equals  $W_{n-2,k-3}$ . Therefore, the number of permutations  $\pi$  in  $\mathfrak{S}_{n,k,1}$  with  $\pi_n = i$  is  $(n-2)W_{n-2,k-3}$ .

From the above discussion, we deduce

$$W_{n,k}^{(1)} = W_{n-1,k}^{(1)} + (n-2)W_{n-2,k-3}. \quad (11)$$

Using Equation (10) for  $W_{n-1,k}^{(1)}$  we further obtain

$$W_{n,k}^{(1)} = W_{n-1,k} - (n-2)W_{n-2,k-1} + (n-2)W_{n-2,k-3},$$

from which we finally recover Equation (3) by replacing the left-hand side using again Equation (10).  $\square$

#### 4.2. Proof of Equation (4)

*Proof.* Let again  $n \geq 1$  and  $3 \leq k \leq \lfloor \frac{3(n-1)}{2} \rfloor$  be fixed. With the same notation as in the previous subsection, and using Equation (10), we see that it suffices to prove

$$W_{n,k}^{(1)} = \sum_{i=1}^{n-2} iW_{i,k-3}. \quad (12)$$

For  $1 \leq i \leq n-2$ , let  $\mathcal{F}_i$  denote the set of permutations  $\pi \in \mathfrak{S}_{n,k,1}$  with  $i+2 = \arg\max_{1 \leq j \leq n} \{\pi_j \neq j\}$ . We thus have  $\pi_{i+2} \neq i+2$  and  $\pi$  fixes all elements from  $i+3$  to  $n$ . Note that  $\max\{j = 1, 2, \dots, n : \pi_j \neq j\} \notin \{1, 2\}$  since we assume  $k \geq 3$ . Therefore,

$$W_{n,k}^{(1)} = \sum_{i=1}^{n-2} |\mathcal{F}_i|. \quad (13)$$

To any permutation  $\pi \in \mathcal{F}_i$ , we can bijectively associate a permutation  $\tau \in \mathfrak{S}_{i+1}$  obtained from  $\pi$  by deleting singletons  $(1), (i+3), (i+4), \dots, (n)$  and renaming element  $i+2$  into 1. The resulting permutation  $\tau$  verifies  $\tau_1 \neq 1$ , and  $\text{perc}(\tau) = k-2$  by Equation (2). Using this bijection and Equation (9), we obtain

$$|\mathcal{F}_i| = W_{i+1, k-2} - W_{i+1, k-2}^{(1)} = iW_{i, k-3},$$

and Equation (13) allows us to complete the proof.  $\square$

We note that Qiu and Akl [16] give an alternative combinatorial proof for Equation (11) and then derive Equation (12) by recurrence.

#### 4.3. Proof of Equation (5)

We give here a combinatorial proof for Equation (5), which was proved by Qiu and Akl [16] by induction, in a direct computational manner.

*Proof.* For every  $1 \leq i \leq k+1$ , we let  $B_i = \mathfrak{S}_{n+k+1, k, n+i}$ . We first prove that

$$W_{n+k+1, k} = |B_1 \cup B_2 \cup \dots \cup B_{k+1}|. \quad (14)$$

To achieve this, we show that any permutation  $\pi \in \mathfrak{S}_{n+k+1, k}$  fixes at least one element among  $n+1, n+2, \dots, n+k+1$ , and therefore  $\pi \in \bigcup_{i=1}^{k+1} B_i$ , which will imply Equation (14).

- If  $\pi_1 = 1$ , from Equation (2) we deduce  $c_1(\pi) = n+1 + c_{\geq 2}(\pi) \geq n+1$ . Therefore, at least one element among  $n+1, n+2, \dots, n+k+1$  must be a singleton.
- If  $\pi_1 \neq 1$ , then Equation (2) implies  $c_1(\pi) = n + c_{\geq 2}(\pi) - 1 \geq n$ . Since 1 is not a singleton, there must also be at least one singleton among the elements  $n+1, n+2, \dots, n+k+1$ .

Since the roles of the elements  $n+1, \dots, n+k+1$  are perfectly interchangeable, we have  $|B_{j_1} \cap B_{j_2} \cap \dots \cap B_{j_i}| = |B_1 \cap B_2 \cap \dots \cap B_i|$ , for every  $1 \leq j_1 < j_2 < \dots < j_i \leq n$  and  $1 \leq i \leq k+1$ . From Equation (14) and the inclusion-exclusion rule, we deduce

$$W_{n+k+1, k} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k+1}{i} |B_1 \cap B_2 \cap \dots \cap B_i|.$$

In order to prove Equation (5), it must be noted that for every  $1 \leq i \leq k+1$ :

$$|B_1 \cap B_2 \cap \dots \cap B_i| = W_{n+k+1-i, k}. \quad (15)$$

Indeed, for every  $1 \leq i \leq k + 1$ , we can define the following bijection,

$$\begin{aligned} \xi_i : B_1 \cap B_2 \cap \cdots \cap B_i &\rightarrow \mathfrak{S}_{n+k+1-i,k} \\ : \pi &\mapsto \tau = \text{red}(\text{dcd}(\pi) \setminus \{(n+1), \dots, (n+i)\}), \end{aligned}$$

which proves Equation (15).  $\square$

## 5. Generating function, mean and variance of the distance distribution

We obtain in this section expressions for the mean  $\mu_n$  and the variance  $\sigma_n^2$  of the prefix exchange distance distribution. More precisely, for a uniform random permutation  $\pi$  in  $\mathfrak{S}_n$ , we have  $\mathbb{P}(\text{pexc}(\pi) = k) = W_{n,k}/n!$  and

$$\begin{aligned} \mu_n &= \mathbb{E}(\text{pexc}(\pi)) = \frac{1}{n!} \sum_{k=0}^{\infty} k W_{n,k}, \\ \sigma_n^2 &= \text{Var}(\text{pexc}(\pi)) = \frac{1}{n!} \sum_{k=0}^{\infty} k^2 W_{n,k} - \mu_n^2. \end{aligned}$$

We start by computing the ordinary generating function

$$W_n(x) = \sum_{k=0}^{\infty} W_{n,k} x^k.$$

As is well-known (see e.g. Wilf [21]), the mean and the variance can be obtained by derivating  $W_n(x)$ :

$$\mu_n = \frac{W'_n(1)}{n!}; \tag{16}$$

$$\sigma_n^2 = \frac{W''_n(1)}{n!} + \mu_n - \mu_n^2 = \frac{W''_n(1)}{n!} - \mu_n(\mu_n - 1). \tag{17}$$

### 5.1. The generating function

We give here an alternative proof of a formula known to Portier and Vaughan [15] for computing the ordinary generating function  $W_n(x)$  of the sequence  $(W_{n,k})_{k \geq 0}$ . Our proof uses Theorem 2.2 as a starting point, whereas Portier and Vaughan (with subsequent corrections by Shen and Qiu [18]) first computed the generating function, then used it to derive the expression in Theorem 2.2.

**Theorem 5.1.** *The ordinary generating function for the prefix exchange distance distribution is given, for every  $x \in \mathbb{C}$ , by the following formula:*

$$W_n(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} (1-x^2)^{n-1-i} x^i \prod_{j=1}^i (x+j). \quad (18)$$

*Proof.* Let  $n \geq 1$  be fixed. Interchanging the order of summation in the formula of Theorem 2.2 yields

$$W_n(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{t=0}^{n-1-i} \binom{n-1-i}{t} \sum_{k=2t+i-1}^{2t+2i} s(i+1, k-i+1-2t) (-1)^{k+2-t} x^k,$$

with the convention  $s(1, -1) = 0$ . Given  $0 \leq i \leq n-1$  and  $0 \leq t \leq n-i-1$ , the bounds on  $k$  come from the conditions  $\lceil \frac{k-2i}{2} \rceil \leq t \leq \lfloor \frac{k+1-i}{2} \rfloor$  appearing in Theorem 2.2. Setting  $j = k - 2t - i + 1$ , we obtain

$$\begin{aligned} & W_n(x) \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{t=0}^{n-1-i} \binom{n-1-i}{t} (-1)^t x^{2t+i-1} \sum_{j=0}^{i+1} s(i+1, j) (-1)^{i+1+j} x^j \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{t=0}^{n-1-i} \binom{n-1-i}{t} (-1)^t x^{2t+i-1} \sum_{j=0}^{i+1} \begin{bmatrix} i+1 \\ j \end{bmatrix} x^j \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{t=0}^{n-1-i} \binom{n-1-i}{t} (-1)^t x^{2t+i-1} x^{\overline{i+1}} \quad (\text{using Equation (1)}) \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} x^{i-1} x^{\overline{i+1}} \sum_{t=0}^{n-1-i} \binom{n-1-i}{t} (-1)^t x^{2t}. \end{aligned}$$

The expression in the statement then follows from Newton's binomial formula, with the convention that  $\prod_{j=1}^0 (x+j) = 1$ .  $\square$

## 5.2. Mean and variance of the distance distribution

Let  $\pi$  be a uniform random permutation in  $\mathfrak{S}_n$ . We will derive expressions for its mean  $\mu_n$  and variance  $\sigma_n^2$  which will involve the  $n$ -th harmonic number  $H_n = \sum_{k=1}^n 1/k$ .

Let  $n \geq 3$ . Using Equation (18), we can write:

$$W_n(x) = \prod_{j=1}^{n-1} [x(x+j)] + (n-1)(1-x^2) \prod_{j=1}^{n-2} [x(x+j)] + g(x) + (1-x^2)^3 h(x), \quad (19)$$

where  $h(x)$  is some polynomial function and

$$g(x) = \frac{(n-1)(n-2)}{2} (1-x^2)^2 \prod_{j=1}^{n-3} [x(x+j)].$$

### 5.2.1. Computation of the mean

We now derive an expression for the expected prefix exchange distance. We note that the value of  $\mu_n$  can be obtained as a particular case of Theorem 6.1, page 203 of Cheng et al. [6] by setting  $k = n - 1$  in the formula they derive. We give here a direct proof of that expression, which provides elements that will prove useful in obtaining the variance of the prefix exchange distance.

**Theorem 5.2.** *Let  $n \geq 1$ . The expected value  $\mu_n$  of the prefix exchange distance for a uniform random permutation in  $\mathfrak{S}_n$  equals*

$$\mu_n = n + H_n - 4 + \frac{2}{n}. \quad (20)$$

*Proof.* We evaluate the derivative of  $W_n(x)$  at  $x = 1$  using the simplified expression in Equation (19). For  $n \geq 3$ , we have

$$\begin{aligned} W'_n(x) &= \sum_{i=1}^{n-1} \frac{2x+i}{x(x+i)} \prod_{j=1}^{n-1} [x(x+j)] + (n-1) \left( -2x + (1-x^2) \sum_{i=1}^{n-2} \frac{2x+i}{x(x+i)} \right) \prod_{j=1}^{n-2} [x(x+j)] \\ &\quad + g'(x) + [(1-x^2)^3 h(x)]'. \end{aligned}$$

When  $x = 1$ , both  $g'(x)$  and  $[(1-x^2)^3 h(x)]'$  vanish, and we obtain:

$$\begin{aligned} W'_n(1) &= n! \sum_{i=1}^{n-1} \frac{i+2}{i+1} - 2(n-1)(n-1)! \\ &= n! \left( n-1 + \sum_{i=1}^{n-1} \frac{1}{i+1} \right) - 2(n-1)(n-1)! \\ &= n!(n + H_n - 2) - 2(n-1)(n-1)!. \end{aligned}$$

Using Equation (16), we deduce

$$\mu_n = n + H_n - 2 - 2 \left( 1 - \frac{1}{n} \right) = n + H_n - 4 + \frac{2}{n}.$$

Note that this expression remains valid for  $n = 1$  and  $n = 2$ ; the above assumption  $n \geq 3$  was forced on us by Equation (19).  $\square$

### 5.2.2. Computation of the variance

We will prove the following:

**Theorem 5.3.** *Let  $n \geq 2$ . The variance  $\sigma_n^2$  of the prefix exchange distance for a uniform random permutation in  $\mathfrak{S}_n$  equals*

$$\sigma_n^2 = H_n + \frac{4}{n} - \frac{8}{n^2} - \sum_{j=1}^n \frac{1}{j^2}. \quad (21)$$

*Proof.* We evaluate the second derivative of  $W_n(x)$  at  $x = 1$ . We first note that we can rewrite the previous expression for  $W'_n(x)$  as

$$W'_n(x) = \left\{ \sum_{i=1}^{n-1} \frac{2x+i}{x(x+i)} - \frac{2(n-1)}{x+n-1} \right\} \prod_{j=1}^{n-1} [x(x+j)] \\ + (n-1)(1-x^2) \sum_{i=1}^{n-2} \frac{2x+i}{x(x+i)} \prod_{j=1}^{n-2} [x(x+j)] + g'(x) + [(1-x^2)^3 h(x)]'.$$

Using the fact that

$$\frac{2x+i}{x(x+i)} = \frac{1}{x} + \frac{1}{x+i}$$

and derivating a second time, we obtain

$$W''_n(x) = \left\{ - \sum_{i=1}^{n-1} \left( \frac{1}{x^2} + \frac{1}{(x+i)^2} \right) + \frac{2(n-1)}{(x+n-1)^2} \right\} \prod_{j=1}^{n-1} [x(x+j)] \\ + \left\{ \sum_{i=1}^{n-1} \left( \frac{1}{x} + \frac{1}{x+i} \right) - \frac{2(n-1)}{x+n-1} \right\} \sum_{k=1}^{n-1} \left( \frac{1}{x} + \frac{1}{x+k} \right) \prod_{j=1}^{n-1} [x(x+j)] \\ - 2(n-1)x \sum_{i=1}^{n-2} \left( \frac{1}{x} + \frac{1}{x+i} \right) \prod_{j=1}^{n-2} [x(x+j)] + (1-x^2)u(x) \\ + g''(x) + [(1-x^2)^3 h(x)]'',$$

where  $u(x)$  is some polynomial function. Since  $[(1-x^2)^3 h(x)]''$  vanishes when  $x = 1$ , we have

$$W''_n(1) = \left\{ - \sum_{i=1}^{n-1} \left( 1 + \frac{1}{(1+i)^2} \right) + \frac{2(n-1)}{n^2} \right\} \prod_{j=1}^{n-1} (1+j) \\ + \left\{ \sum_{i=1}^{n-1} \left( 1 + \frac{1}{1+i} \right) - \frac{2(n-1)}{n} \right\} \sum_{k=1}^{n-1} \left( 1 + \frac{1}{1+k} \right) \prod_{j=1}^{n-1} (1+j) \\ - 2(n-1) \sum_{i=1}^{n-2} \left( 1 + \frac{1}{1+i} \right) \prod_{j=1}^{n-2} (1+j).$$

Replacing  $\prod_{j=1}^{n-1} (1+j)$  with  $n!$  and  $\sum_{i=1}^{n-1} \left( 1 + \frac{1}{1+i} \right)$  with  $n + H_n - 2$  yields

$$W''_n(1) = n! \left\{ -n + 2 + \frac{2}{n} - \frac{2}{n^2} - \sum_{j=1}^n \frac{1}{j^2} + \left( n + H_n - 4 + \frac{2}{n} \right) (n + H_n - 2) \right\} \\ - 2(n-1)(n-1)! \left( n + H_n - 3 - \frac{1}{n} \right) + g''(1).$$

We must now compute  $g''(1)$ . We have, with  $f(x)$  being some polynomial function:

$$g'(x) = 2(n-1)(n-2) \left\{ -(1-x^2)x^{n-2} \prod_{j=1}^{n-3} (x+j) + (1-x^2)^2 f(x) \right\}.$$

When derivating a second time and taking  $x = 1$  we obtain

$$g''(1) = 4(n-2)(n-1)!.$$

Injecting this expression in the previous formula for  $W_n''(1)$  and dividing by  $n!$  gives

$$\begin{aligned} \frac{W_n''(1)}{n!} &= -n + 2 + \frac{2}{n} - \frac{2}{n^2} - \sum_{j=1}^n \frac{1}{j^2} + \left( n + H_n - 4 + \frac{2}{n} \right) (n + H_n - 2) \\ &\quad - 2 \left( 1 - \frac{1}{n} \right) \left( n + H_n - 3 - \frac{1}{n} \right) + 4 - \frac{8}{n}. \end{aligned}$$

Using Equation (17), we obtain that the variance of the distance distribution equals

$$\begin{aligned} \sigma_n^2 &= -n + 6 - \frac{6}{n} - \frac{2}{n^2} - \sum_{j=1}^n \frac{1}{j^2} + \left( n + H_n - 4 + \frac{2}{n} \right) (n + H_n - 2) \\ &\quad - 2 \left( 1 - \frac{1}{n} \right) \left( n + H_n - 3 - \frac{1}{n} \right) - \left( n + H_n - 4 + \frac{2}{n} \right) \left( n + H_n - 5 + \frac{2}{n} \right) \\ &= -n + 6 - \frac{6}{n} - \frac{2}{n^2} - \sum_{j=1}^n \frac{1}{j^2} + \left( n + H_n - 4 + \frac{2}{n} \right) \left( 3 - \frac{2}{n} \right) \\ &\quad - 2 \left( 1 - \frac{1}{n} \right) \left( n + H_n - 3 - \frac{1}{n} \right), \end{aligned}$$

which finally gives the formula in Equation (21).  $\square$

## 6. Asymptotic behaviour of the distance distribution

**Proposition 6.1.** *We have the following asymptotics for the mean and the variance of the prefix exchange distribution when  $n$  is large:*

$$\begin{aligned} \mu_n &= n + \log n + \gamma - 4 + o(1); \\ \sigma_n^2 &= \log n + \gamma - \frac{\pi^2}{6} + o(1), \end{aligned}$$

where  $\gamma \approx 0.577$  is the Euler-Mascheroni constant and  $o(1)$  denotes a sequence converging to 0 as  $n \rightarrow \infty$ .



*Proof.* Immediate from Equations (20) and (21), using the well-known results (see e.g. Graham et al. [10]):

$$H_n - \log n \longrightarrow \gamma \text{ when } n \rightarrow \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

We further show that, for large  $n$ , the distribution of the prefix exchange distance for a uniform random permutation  $\pi \in \mathfrak{S}_n$  is approximately normal, with mean  $\mu_n$  and variance  $\sigma_n^2$ . More precisely, we prove the following:

**Theorem 6.1.** *The normalised prefix exchange distance for a uniform random permutation  $\pi \in \mathfrak{S}_n$ , i.e.*

$$D_n = \frac{\text{pexc}(\pi) - \mu_n}{\sigma_n}$$

*converges in distribution, when  $n \rightarrow \infty$ , to the standard normal distribution  $\mathcal{N}(0, 1)$ , which means that*

$$\mathbb{P}(a < D_n < b) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

*when  $n \rightarrow \infty$ , for every real numbers  $a < b$ .*

**Remark 6.1.** *Using the asymptotics for  $\mu_n$  and  $\sigma_n^2$  derived in Proposition 6.1, the above convergence is equivalent to the following convergence in distribution*

$$\frac{\text{pexc}(\pi) - n - \log n}{\sqrt{\log n}} \longrightarrow \mathcal{N}(0, 1) \text{ when } n \rightarrow \infty,$$

*which means that the distribution of the prefix exchange distance for a uniform random permutation  $\pi \in \mathfrak{S}_n$  is asymptotically normal, with mean  $n + \log n$  and variance  $\log n$ .*

*Proof of Theorem 6.1.* We will show that the sequence of characteristic functions of the random variables  $(D_n)_{n \geq 1}$  converges pointwise, when  $n \rightarrow \infty$ , to the characteristic function of the standard normal distribution, given by  $\varphi(t) = e^{-t^2/2}$ . Lévy's convergence theorem (see e.g. Billingsley [3]) will then imply that the sequence  $(D_n)_{n \geq 1}$  converges in distribution to the standard normal distribution  $\mathcal{N}(0, 1)$ .

Let  $\varphi_n$  denote the characteristic function of the random variable  $D_n$ , defined for  $t \in \mathbb{R}$  by  $\varphi_n(t) = \mathbb{E}(e^{itD_n})$ . We have

$$\varphi_n(t) = e^{-\frac{it\mu_n}{\sigma_n}} \sum_{k=0}^{\infty} e^{\frac{itk}{\sigma_n}} \mathbb{P}(\text{pexc}(\pi) = k).$$

Since  $\pi$  is chosen uniformly at random in  $\mathfrak{S}_n$ , we have

$$\mathbb{P}(\text{prec}(\pi) = k) = \frac{W_{n,k}}{n!},$$

which yields

$$\varphi_n(t) = e^{-\frac{it\mu_n}{\sigma_n}} \frac{W_n(e^{\frac{it}{\sigma_n}})}{n!}, \quad (22)$$

where  $W_n(\cdot)$  is the generating function obtained in Equation (18). For every  $x \in \mathbb{C}$ , Equation (18) reads

$$W_n(x) = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!} (1-x^2)^{n-1-k} x^k \prod_{j=1}^k (x+j). \quad (23)$$

Equations (22) and (23) then yield, for any  $t \in \mathbb{R}$ :

$$\varphi_n(t) = e^{-\frac{it\mu_n}{\sigma_n}} \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{itk}{\sigma_n}} \frac{(1 - e^{\frac{2it}{\sigma_n}})^{n-k-1}}{(n-k-1)!} \frac{\prod_{j=1}^k (e^{\frac{it}{\sigma_n}} + j)}{k!}.$$

We will show that the dominant term is obtained for  $k = n-1$  and converges to  $e^{-t^2/2}$  when  $n \rightarrow \infty$ , while all other terms vanish at the limit. To that end, let us isolate in  $\varphi_n(t)$  the term obtained for  $k = n-1$  (which we denote  $A_n$ ) and let  $R_n$  denote the sum of all other terms; we obtain:

$$\varphi_n(t) = A_n + R_n, \quad (24)$$

with

$$A_n = \exp\left(-\frac{it(\mu_n - n + 1)}{\sigma_n}\right) \frac{\prod_{j=1}^{n-1} (e^{\frac{it}{\sigma_n}} + j)}{n!} \quad (25)$$

and

$$|R_n| \leq \frac{1}{n} \sum_{k=0}^{n-2} \frac{|1 - e^{\frac{2it}{\sigma_n}}|^{n-k-1}}{(n-k-1)!} \frac{\prod_{j=1}^k (1+j)}{k!} = \sum_{k=0}^{n-2} \frac{k+1}{n} \frac{|1 - e^{\frac{2it}{\sigma_n}}|^{n-k-1}}{(n-k-1)!},$$

using the fact that  $|e^{ix}| = 1$ , for  $x \in \mathbb{R}$ .

Let us first show that  $R_n$  converges to 0 when  $n \rightarrow \infty$ . Setting  $j = n-k-1$  in the above inequality, we obtain:

$$|R_n| \leq \sum_{j=1}^{n-1} \frac{|1 - e^{\frac{2it}{\sigma_n}}|^j}{j!} \leq \sum_{j=1}^{\infty} \frac{|1 - e^{\frac{2it}{\sigma_n}}|^j}{j!} = \exp(|1 - e^{\frac{2it}{\sigma_n}}|) - 1,$$

using the MacLaurin series, and therefore  $R_n \rightarrow 0$  as  $n \rightarrow \infty$ .

To show that  $\varphi_n(t) \rightarrow e^{-t^2/2}$ , in light of Equation (24), we must check that

$$A_n = \exp\left(-\frac{it(\mu_n - n + 1)}{\sigma_n}\right) \frac{\prod_{j=1}^{n-1} (e^{\frac{it}{\sigma_n}} + j)}{n!} \rightarrow e^{-t^2/2}.$$

Note that the product  $\prod_{j=1}^{n-1} (e^{\frac{it}{\sigma_n}} + j)$  can be written as a ratio of two Gamma functions. We recall that for  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > 0$ , the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

By integration by parts, it is easy to see that the Gamma function satisfies the recurrence relation  $\Gamma(z+1) = z\Gamma(z)$ , which implies, in particular, that for  $n \in \mathbb{N}^*$  we have  $\Gamma(n) = (n-1)!$ .

The same recurrence relation allows us to write:

$$\prod_{j=1}^{n-1} (e^{\frac{it}{\sigma_n}} + j) = \frac{\Gamma(n + e^{\frac{it}{\sigma_n}})}{\Gamma(e^{\frac{it}{\sigma_n}})},$$

for  $n$  sufficiently large to have  $\operatorname{Re}(e^{\frac{it}{\sigma_n}}) > 0$ .

We further use the following asymptotic approximation (see e.g. Tricomi and Erdélyi [20]):

$$\frac{\Gamma(n + e^{ix})}{n!} = n^{e^{ix}-1} (1 + o(1)),$$

for  $x \in \mathbb{R}$  and  $n = 2, 3, \dots$ , to deduce

$$\frac{\prod_{j=1}^{n-1} (e^{\frac{it}{\sigma_n}} + j)}{n!} = \frac{\Gamma(n + e^{\frac{it}{\sigma_n}})}{n! \Gamma(e^{\frac{it}{\sigma_n}})} = \frac{n^{e^{\frac{it}{\sigma_n}}-1}}{\Gamma(e^{\frac{it}{\sigma_n}})} (1 + o(1)).$$

As a consequence, and based on the asymptotic approximation of  $\mu_n$  from Proposition 6.1, we deduce from Equation (25) that for  $n \rightarrow \infty$ :

$$A_n = \exp\left(-\frac{it \log n}{\sigma_n}\right) \frac{n^{e^{\frac{it}{\sigma_n}}-1}}{\Gamma(e^{\frac{it}{\sigma_n}})} (1 + o(1)). \quad (26)$$

We further write

$$n^{e^{\frac{it}{\sigma_n}}-1} = \exp\left(\log\left(n^{e^{\frac{it}{\sigma_n}}-1}\right)\right) = \exp\left(\left(e^{\frac{it}{\sigma_n}} - 1\right) \log n\right).$$

The second order series expansion of the exponential, together with the asymptotic approximation of  $\sigma_n^2$  from Proposition 6.1 yield

$$e^{\frac{it}{\sigma_n}} - 1 = \frac{it}{\sigma_n} - \frac{t^2}{2\sigma_n^2} + o\left(\frac{1}{\log n}\right),$$

and

$$n^{e^{\frac{it}{\sigma_n}}-1} = \exp\left(\frac{it \log n}{\sigma_n} - \frac{t^2}{2}\right) (1 + o(1)).$$

Further replacing in Equation (26) implies

$$A_n = \exp\left(-\frac{t^2}{2}\right) (1 + o(1)).$$

We have also used the fact that, by continuity, the denominator  $\Gamma(e^{\frac{it}{\sigma_n}})$  converges to  $\Gamma(1) = 1$  as  $n \rightarrow \infty$ . Since  $\varphi_n(t) = A_n + R_n$  and  $R_n \rightarrow 0$ , it finally follows that  $\varphi_n(t)$  converges to  $e^{-t^2/2}$  as  $n \rightarrow \infty$ , which ends the proof.  $\square$

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