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Stress-gradient materials: an analytical exploration

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Catania, 29-31 October, 2015
Motivation (1)

At first glance...

Strain gradient elasticity theory

\[ w = w(\varepsilon, \nabla\varepsilon) \]

strain energy density

strain

strain gradient

Motivation (2)

At first glance...

Strain gradient elasticity theory

\[ w = w(\varepsilon, \nabla \varepsilon) \]

Stress gradient elasticity theory

\[ w^* = w^*(\sigma, \nabla \sigma) \]

S. Forest, K. Sab
Mechanics Research Communications, 40, pp. 16–25, 2012

V. P. Tran, S. Brisard, J. Guilleminot, K. Sab

Motivation

Catania, 29-31 October, 2015
Motivation (3)

At first glance...

Strain gradient elasticity theory

\[ w = w(\varepsilon, \nabla\varepsilon) \]

Stress gradient elasticity theory

\[ w^* = w^*(\sigma, \nabla\sigma) \]

Is stress gradient elasticity equivalent or complementary to strain gradient elasticity?
How does stress gradient elasticity theory (Forest and Sab, 2012) differ from strain gradient one?

- Stress gradient elasticity theory (Forest and Sab, 2012).
- Closed form solution to Eshelby spherical inhomogeneity problem.
- Homogenization of heterogeneous stress-gradient materials (Mori Tanaka estimate)
Decomposition of stress gradient tensor

The spherical and deviatoric part of a third order tensor

**Strain gradient tensor**
- "free"

**Stress gradient tensor**
- constrained

\[
\nabla \cdot \sigma + b = 0 \iff (\nabla \sigma) : \delta + b = 0
\]

**Decomposition of stress gradient tensor**

\[
\nabla \sigma = J \cdot \nabla \sigma + K \cdot \nabla \sigma
\]

spherical projector
deviatoric projector

\[
J_{ijklmn} = \frac{1}{8} (\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln})
\]

\[
K_{ijklmn} = I_{ijklmn} - J_{ijklmn}
\]
**Stress energy density function**

\[ w^* = w^*(\sigma, \text{K} \cdot \nabla \sigma) \]

For isotropic stress gradient materials:

\[ w^*(\sigma, R) = \frac{1}{2} \sigma : S : \sigma + \frac{1}{2} R : M : R \]

**Constitutive laws**

\[ e = \frac{\partial w^*}{\partial \sigma} = S : \sigma, \quad \Phi = \frac{\partial w^*}{\partial R} = M : R \]

**Nota:** \( \Phi, R \) have to be deviatoric tensors!

\[ M = K \cdot M : K \]
# Simplified stress / strain gradient elasticity

A comparison of field equations (no body force)

<table>
<thead>
<tr>
<th></th>
<th>Simplified stress gradient</th>
<th>Simplified strain gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equilibrium</strong></td>
<td>$\sigma_{ij,j} = 0$</td>
<td>$\tau_{ij,j} = 0$</td>
</tr>
<tr>
<td><strong>Gradient</strong></td>
<td>$R_{ijk} = \sigma_{ij,k}$</td>
<td>$\kappa_{ijk} = \varepsilon_{ij,k}$</td>
</tr>
<tr>
<td><strong>Constitutive laws</strong></td>
<td>$e_{ij} = S_{ijkl} \sigma_{kl}$</td>
<td>$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$</td>
</tr>
<tr>
<td></td>
<td>$e_{ij} = \varepsilon_{ij} + \Phi_{ijk,k}$</td>
<td>$\tau_{ij} = \sigma_{ij} - \mu_{ijk,k}$</td>
</tr>
<tr>
<td></td>
<td>$\Phi_{ijk} = \ell^2 K_{ijkpqr} S_{pqmn} R_{mnr}$</td>
<td>$\mu_{ijk} = \ell^2 C_{ijmn} \kappa_{mnk}$</td>
</tr>
<tr>
<td><strong>SUBC</strong></td>
<td>$\sigma_{ij} = \Sigma_{ij}$</td>
<td>$\sigma_{ij} n_j - (\mu_{ijk} n_k)<em>j + (\mu</em>{ijk} n_k n_l)_l n_j = t_i$ on $\partial \Omega$ (smooth part)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mu_{ijk} n_j n_k = q_i$ on $\partial \Omega$ (smooth part)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$[[\mu_{ijk} m_j n_k]] = 0$ on the edge of $\partial \Omega$</td>
</tr>
</tbody>
</table>

S. Forest, K. Sab, Mechanics Research Communications, 40, pp. 16–25, 2012

V.P. Tran, S. Brisard, J. Guilleminot, K. Sab, Simplified stress / strain gradient elasticity, Catania, 29-31 October, 2015
Stress gradient (this work):  
- Uniform isotropic stress:  
  \[ \Sigma = e_x \otimes e_x + e_y \otimes e_y + e_z \otimes e_z \]
- Uniform axial stress:  
  \[ \Sigma = e_z \otimes e_z \]

Strain gradient:
Postulated stress form in spherical coordinates (similar to Love’s approach)

- Uniform isotropic stress prescribed at boundary:
  \[ \sigma = \sigma(r) \]
- Uniform axial stress prescribed at boundary:
  \[ \sigma = \sigma(r, \theta) \]

Continuity conditions at the surface matrix-inclusion

\[ \sigma_m(r = a) = \sigma_i(r = a) \]
\[ [u \otimes^s n + \Phi \cdot n]_m (r = a) = [u \otimes^s n + \Phi \cdot n]_i (r = a) \]

Uniaxial problem $\Sigma = e_z \otimes e_z$

Radial stress field $\sigma_{rr}(r, \theta = 0)$

$$\mu_i = 27\mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell$$

Classical elasticity
Stress gradient

$\sigma_{rr}(r, \theta = 0) / \Sigma$

$\ell/a = 0$
$\ell/a = 1$

$r/a$

$\ell/a = 0$
$\ell/a = 1$
Effect of material internal length (1)

Effective shear moduli (Mori Tanaka estimate)

\[ \mu_i = 27\mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell \]

Stress gradient is complementary to strain gradient elasticity!

Effect of material internal length (2)

Effective shear moduli: \( f = 20\% \)

\[
\mu_i = 27 \mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell
\]

- **Stress gradient elasticity:** capture size effects.
- **For** \( a \gg \ell \), stress gradient effects vanish as expected.
Softening size effect

... has been reported elsewhere (molecular dynamics simulations)!

To cite some:

- Polyimide matrix "reinforced" by silica spherical nanoparticles with various surface treatment.
  
  Odegard, G.M., Clancy, T.C., Gates, T.S. Polymer, 46, pp.553–562, 2005

- Polystyrene matrix "reinforced" by silica spherical nanoparticles.
  

- etc
Conclusion & Perspective

Conclusion

- Solution to Eshelby’s spherical inhomogeneity problem
- Mori Tanaka estimate for stress gradient materials
- Stress gradient elasticity differs from strain gradient elasticity
  - Strain gradient: stiffening size effect
  - Stress gradient: softening size effect

Perspective

- Numerical implementation: FEM, FFT
- Physical interpretation of material internal length?

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Thank you for your attention!

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