A thermodynamic approach to effective stresses in unsaturated soils incorporating the concept of partial pore deformations

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Abstract

The thermodynamical analysis presented here follows from the work of Coussy et al [13] who proposed a thermodynamically consistent model for unsaturated soils which is based on a Bishop-like effective stress to describe the stress-strain relationship while the water saturation (or the capillary pressure) is involved in a saturation-induced hardening in addition to the mechanical hardening. We extended this model to include the effect of interfaces in the mechanical behaviour and we showed that the Bishop-like stresses involved in the elastic and plastic responses respectively can take different expressions. The Modified Cam-Clay model used for saturated soils is extended to unsaturated soils through the use of

these Bishop-like stresses. This model is compared to some experimental results reported from the literature.

Keywords: Unsaturated soil, Effective stress, Thermodynamics

1 Introduction

The concept of effective stress in unsaturated soils goes back to the work of Bishop [4] who extended the concept of Terzaghi's effective stress by introducing a weighted average of gas and liquid pressures [5, 7]. This proposal encountered difficulties in explaining collapse behaviour [6, 9, 1, 23]. Then many authors have pointed out the need of two independent stress state variables to account for experimental observations on the mechanical behavior of unsaturated soils [18]. On that basis elastoplastic models were formulated [2, 20]. These models can be viewed as an extension of the Cam-Clay model to unsaturated situations. This has launched the development of many other models [24, 30, 7, 28, 29, 19, 36, 34, 15, 33]. All these models are founded on two independent stresses even though they vary widely in the choice of the stresses. Some of them [32] chose to refer one stress to a Bishop-type stress. But all those models require the suction or the capillary pressure as an additional and independent stress. The reader can refer to the comprehensive review of effective stresses proposed by Nuth et al [31]. In most of these models suction is a hardening variable and thus has a status somehow different from the Bishop stress. As noted by Coussy [13, 12] the status of the suction or capillary pressure is two fold. Its variations control

the fluid invasion process through the retention curve and they also control the mechanical behaviour through the deformation of the pores they induced. This can be a source of confusion in the formulation of the constitutive equations as pointed out by Alonso et al [3]. A significant breakthrough in the way of clarification, was achieved by Coussy [12] who proposed a more appropriate definition of the saturation degree, called Lagrangian saturation degree. This new definition is only associated to the invasion process, i.e to the creation and destruction of fluid-solid interface areas during wetting-drying processes. In contrast this saturation degree is not affected by the deformation process of the porous network. Thanks to this new concept Coussy et al [13] have given a physical background to the coefficient involved in the Bishop effective stress and proposed, on this physical basis, an extension of the Cam-Clay model to unsaturated conditions which is thermodynamically consistent. Experimental data on shear strength suggest that this Bishop coefficient is mostly smaller than the saturation degree generally used in the expression of the Bishop effective stress [3].

Following the approach of Coussy, we explore here the effect of the interface energy on the mechanical behaviour of unsaturated soils which was neglected in the work of Coussy [11, 13]. As opposed to what was done in Coussy, the interface energy here depends on the deformation of the material. We also derive two Bishop-like effective stresses related to the elastic and plastic responses respectively. Finally we propose a simple extension of the Modified Cam-Clay model to unsaturated conditions and some comparisons with experimental data

are shown.

An unsaturated soil consists in a solid skeleton composed of solid grains in contact, a gas phase and a liquid phase. These three phases interact with each other through interfaces which sustain surface stress and possess their own energy. These interfaces play a fondamental role in the thermodynamic analysis of unsaturated soils. Accordingly the thermodynamics of a representative volume element of unsaturated soil can be addressed by considering three different systems. The first one is the soil itself, as depicted above, including all the matter in all form contained in the RVE. It is an open thermodynamic system exchanging gas and liquid mass. The second system is obtained by removing the bulk fluid masses whatever the fluid form. It is then formed of the solid phase and the interfaces only. This system is still subjected to the gas and liquid pressures through the interfaces. However these pressure are considered now as external forces. Like Coussy [13] we'll call this system the "apparent solid skeleton" (subscript "ske") since it includes interfaces with energy concentrated on those surfaces. By removing the interfaces we can obtain a third system consisting in only the solid phase. We will call it the "solid matrix" (subscript "sol") in the following. This system is now subjected to external forces which differ from the gas and liquid pressures since part of these pressures are absorbed by the interface surface stresses. We will assume that these forces can be represented by two effective pressures exerted on the part of the solid wall in contact with the solid-liquid and solid-gas interfaces. We will denote them by π_{L} and π_{G}. We must note that such effective pressures have already been derived formally

by other authors from a microscopical approach and by making use of averaging technics [21, 22].

2 Effective pore pressures and interface energy

Consider a volume V_0 of soil in its undeformed reference configuration. In the current configuration the volume is V, the pore volume is ϕV_0 where ϕ is the Lagrangian porosity [11]. The pore volume occupied by the liquid and gas phase are given by $\phi_{\rm L} V_0$ and $\phi_{\rm G} V_0$, where the $\phi_{\rm J}$ (J = L, G) can be coined as partial Lagrangian porosities respectively. Furthermore we have $\phi_{\rm L} + \phi_{\rm G} = \phi$.

The balance free energy of the apparent solid skeleton, at constant temperature, can be expressed as [13]

$$dF_{\rm ske} = \sigma_{ij} d\epsilon_{ij} + p_{\rm L} d\phi_{\rm L} + p_{\rm G} d\phi_{\rm G} \tag{1}$$

The current partial porosity $\phi_{\rm J}$ can be written in the form [12]

$$\phi_{\rm J} = \phi_0 S_{\rm J} + \varphi_{\rm J} \; ; \qquad S_{\rm L} + S_{\rm G} = 1 \tag{2}$$

where $S_{\rm J}$ is the Lagrangian saturation degree and $\varphi_{\rm J}$ is the deformation of the porous network occupied by the phase J which can be coined as the partial pore deformation. In Eq. (2) $\phi_0 S_{\rm J}$ is the volume occupied by the fluid J prior to any deformation i.e. by the part of the porous volume of the undeformed reference configuration which is delimited by the internal solid walls wetted by the fluid J [12]. The variations of $S_{\rm J}$ is therefore associated to the invasion process of interfaces i.e. to the displacement of the common line between the

three interfaces onto the solid surface. Substituting expression (2) for ϕ_J in (1) reads

$$dF_{ske} = \sigma_{ij} d\epsilon_{ij} + p_L d\varphi_L + p_G d\varphi_G - \phi_0 (p_G - p_L) dS_L \tag{3}$$

The three first terms of the right hand side of Eq. (3) represent the deformation work undergone by the apparent solid skeleton while the fourth term is the energy supply to create new or suppress existing inner interfaces. As a consequence the energy of the solid skeleton can be split in two parts:

$$F_{ske} = F_{sol}(\epsilon_{ij}, \varphi_L, \varphi_G, S_L) + F_{int}(\varphi_L, \varphi_G, S_L) \tag{4}$$

where F_{sol} is the free energy stored in the solid matrix and F_{int} is the free energy of interfaces. The free energy of the solid matrix, F_{sol}, is mainly a function of the deformation variables $\epsilon_{ij}, \varphi_L, \varphi_G$ with S_L intervening as a coupling term. In this sense the derivative $\frac{\partial F_{sol}}{\partial S_L}$ will always be coupled with the deformation variables and therefore will be considered as a small term compared to $\frac{\partial F_{int}}{\partial S_L}$. Similarly F_{int} depends essentially on S_L and the partial deformation of pores, φ_L and φ_G, as coupling terms. Because interfaces are located in the porosity, F_{int} is assumed as independent of the skeleton strains.

According to Eq. (3) the force-like vector formed by the stress tensor, the fluid pressures and the pressure difference $-\phi_0(p_G - p_L)$ is energy conjugate to the deformation-like vector formed by the strain tensor, the partial pore deformations and the saturation degree. As already noted by Coussy et al [13]: "In the familiar capillary case, although the suction can be defined as the difference between the pressures of the non-wetting and wetting phases, the

various role of the pressure difference must be well separated from that of the suction". Indeed the status of fluid pressures in the energy change is two fold. On one hand the mechanical pressures that are exerted on complementary parts of the solid wall from the liquid and gas, govern the process of deformation of the material. The saturation degree which controls the partition of these pressures on the solid wall can be considered as an independent parameter of the behaviour and therefore decoupled from these mechanical pressures. On the other hand the process of invasion, linked to the change of saturation degree, is controlled by the suction through the retention curve. Even though the suction is eventually given by the difference between the gas and liquid pressures, the status of the suction is here well separated from that of the mechanical pressures.

From a thermodynamical point of view these different status of fluid pressures form three independant forces which are energy conjugate to the three independent thermodynamical variables: $\varphi_{\mathrm{L}}, \varphi_{\mathrm{G}}, S_{\mathrm{L}}$.

2.1 Energy of the solid matrix

Combining (3) and (4), the free energy of the sole solid matrix satisfies

$$(dF_{\mathrm{sol}})_{S_{\mathrm{L}}} = \sigma_{ij} d\epsilon_{ij} + \pi_{\mathrm{L}} d\varphi_{\mathrm{L}} + \pi_{\mathrm{G}} d\varphi_{\mathrm{G}} \tag{5}$$

where

$$\pi_{\mathrm{J}} = p_{\mathrm{J}} - \left(\frac{\partial F_{\mathrm{int}}}{\partial \varphi_{\mathrm{J}}} \right)_{S_{\mathrm{L}}, \varphi_{\mathrm{K} \neq \mathrm{J}}} \tag{6}$$

The interpretation of the effective pressure π_{J} can be addressed equivalently as follows:

(i) $\pi_{\mathrm{J}} d\varphi_{\mathrm{J}}$ is the infinitesimal deformation work given to the solid matrix through the partial pore deformation $d\varphi_{\mathrm{J}}$.

(ii) π_{J} represents, at the macroscopic scale, the modeling of the actual normal stress exerted to the solid matrix. Therefore π_{J} can be coined as an effective pore pressure.

(iii) $p_{\mathrm{J}} - \pi_{\mathrm{J}} = \frac{\partial F_{\mathrm{int}}}{\partial \varphi_{\mathrm{J}}}$ is due to the surface tension sustained by the solid-fluid interface and can be compared with the Young-Lapace equation[1].

From the balance energy (5) the state laws read

$$\sigma_{ij} = \left(\frac{\partial F_{\mathrm{sol}}}{\partial \epsilon_{ij}} \right)_{S_{\mathrm{L}}, \varphi_{\mathrm{J}}} \qquad \pi_{\mathrm{J}} = \left(\frac{\partial F_{\mathrm{sol}}}{\partial \varphi_{\mathrm{J}}} \right)_{S_{\mathrm{L}}, \epsilon_{ij}, \varphi_{\mathrm{K} \neq \mathrm{J}}} \tag{7}$$

At constant saturation degree S_{L}, the linearization of the state laws (7) can provide a first approach of the constitutive equations of unsaturated soils. However the coefficients involved in the linearization process must depend on S_{L}. As a general rule the variable S_{L} appearing in the arguments of F_{sol} should be considered as a coupling term only. As a consequence expression for F_{sol} should involve only small terms as strains and partial pore deformations: ϵ_{ij}, φ_{J}.

On the other way, a general requirement for F_{sol} can be expressed as follows. Along any loading path characterized by $\pi_{\mathrm{L}} = \pi_{\mathrm{G}}$ the solid wall is subjected to a uniform pore pressure. In that case, according to Eq. (5), we could expect an expression for F_{sol} which is independent of S_{L} as long as $\pi_{\mathrm{L}} = \pi_{\mathrm{G}}$. In other

[1] In case of a spherical pore of radius r, it is easy to show that $\frac{\partial F_{\mathrm{int}}}{\partial \varphi_{\mathrm{J}}} = \frac{2\gamma}{r}$ where γ is the surface tension

words the derivative $\frac{\partial F_{\rm sol}}{\partial S_{\rm L}}$ should vanish for any value $\pi_{\rm L} = \pi_{\rm G}$:

$$\left(\frac{\partial F_{\rm sol}}{\partial S_{\rm L}}\right)_{\epsilon_{ij},\varphi_{\rm K}} = 0 \qquad \forall\ \pi_{\rm L} = \pi_{\rm G} \tag{8}$$

2.2 Energy of interfaces

The interface free energy can be derived from the surface tension, $\gamma_{\rm IJ}$, and the surface area, $\omega_{\rm IJ}$, of the three interfaces according to

$$F_{\rm int} = \gamma_{\rm SL}\omega_{\rm SL} + \gamma_{\rm SG}\omega_{\rm SG} + \gamma_{\rm GL}\omega_{\rm GL} \tag{9}$$

Since the previous approach has postulated that, at the macroscopic scale, this energy only depends on the 3 thermodynamical variables, $(\varphi_{\rm L}, \varphi_{\rm G}, S_{\rm L})$, such an expression must be consistent with expression (9) for any deformation process. To go further we are going to make some reasonable assumptions for interface energy. Surface tensions are assumed constant or only temperature dependent. Accordingly since interface energy is spread over surfaces we can assume the following property regarding the dependence of $F_{\rm int}$ upon the partial pore deformations:

$$F_{\rm int}(\varphi_{\rm L} + \lambda\phi_0 S_{\rm L}, \varphi_{\rm G} + \lambda\phi_0 S_{\rm G}, S_{\rm L}) = (1 + \frac{2}{3}\lambda)F_{\rm int}(\varphi_{\rm L}, \varphi_{\rm G}, S_{\rm L}) \qquad \forall\lambda \ll 1 \tag{10}$$

Statement (10) expresses that the interface energy change, at constant saturation, is only due to the change of the surface areas of pores in the process of deformation. Indeed from the current state any virtual (small) homogeneous dilation of coefficient $(1 + \frac{1}{3}\lambda)$ would cause an increase of volume by a factor $(1+\lambda)$ (i.e the volume of pore $\phi_0 S_{\rm J}$ would increase by $\lambda\phi_0 S_{\rm J}$) and an increase of surface by a factor $(1 + \frac{2}{3}\lambda)$. Property (10) implicitly assumes that the surface

stresses don't depend on the solid surface strains and coincide with the surface tensions, γ_{IJ}, thereby assuming that they are constant. Derivation with respect to λ entails

$$\phi_0 S_{\mathrm{L}} \left(\frac{\partial F_{\mathrm{int}}}{\partial \varphi_{\mathrm{L}}} \right)_{\epsilon_{ij},\varphi_{\mathrm{G}}} + \phi_0 S_{\mathrm{G}} \left(\frac{\partial F_{\mathrm{int}}}{\partial \varphi_{\mathrm{G}}} \right)_{\epsilon_{ij},\varphi_{\mathrm{L}}} = \frac{2}{3} F_{\mathrm{int}} \tag{11}$$

Finally linearizing F_{int} with respect to the partial pore deformations φ_{J} gives

$$F_{\mathrm{int}} = \frac{2}{3} U_{\mathrm{L}}(S_{\mathrm{L}})\varphi_{\mathrm{L}} + \frac{2}{3} U_{\mathrm{G}}(S_{\mathrm{L}})\varphi_{\mathrm{G}} + \phi_0 U(S_{\mathrm{L}}) \tag{12}$$

where U is the interface energy per unit of porous space prior to any deformation process. Combining (11) and (12) shows that U is expressed as

$$U = S_{\mathrm{L}} U_{\mathrm{L}} + S_{\mathrm{G}} U_{\mathrm{G}} \tag{13}$$

where U_{L} and U_{G} are two interface energies associated to the liquid and gas phases. From (6) we have

$$\pi_{\mathrm{J}} = p_{\mathrm{J}} - \frac{2}{3} U_{\mathrm{J}} \tag{14}$$

With the help of the equations derived above the interface energy balance writes

$$dF_{\mathrm{int}} = (p_{\mathrm{L}} - \pi_{\mathrm{L}}) d\varphi_{\mathrm{L}} + (p_{\mathrm{G}} - \pi_{\mathrm{G}}) d\varphi_{\mathrm{G}} - \left(\phi_0 (p_{\mathrm{G}} - p_{\mathrm{L}}) + \frac{\partial F_{\mathrm{sol}}}{\partial S_{\mathrm{L}}} \right) dS_{\mathrm{L}} \tag{15}$$

Neglecting $\frac{\partial F_{\mathrm{sol}}}{\partial S_{\mathrm{L}}}$ compared to $\frac{\partial F_{\mathrm{int}}}{\partial S_{\mathrm{L}}}$, the state laws of interface now read at the first order

$$p_{\mathrm{G}} - p_{\mathrm{L}} = -\frac{dU}{dS_{\mathrm{L}}} \; ; \qquad p_{\mathrm{L}} - \pi_{\mathrm{L}} = \frac{2}{3} U_{\mathrm{L}} \; ; \qquad p_{\mathrm{G}} - \pi_{\mathrm{G}} = \frac{2}{3} U_{\mathrm{G}} \tag{16}$$

These 3 state laws can be compared, in some way, with a kind of macroscopic Young-Laplace law. The first law (16a) is the well known capillary or retention

cure. The two other laws (16b,16c) are unusual and difficult to apply because it is not possible to measure energies $U_{\rm J}$ separately. One possible way to overcome this difficulty comes from an exploitation of the microscopic definition of the interface energy (9) which can be written, in the undeformed state and using the Young equation, as

$$\phi_0 U = \gamma_{\rm SL}\left(\omega_{\rm SL} - \frac{\omega_{\rm GL}}{\cos\theta}\right) + \gamma_{\rm SG}\left(\omega_{\rm SG} + \frac{\omega_{\rm GL}}{\cos\theta}\right) \tag{17}$$

where θ is the contact angle of the liquid assumed as the wetting phase. Then we assume that each term of the rhs of Eq. (17) can be identified to that of the rhs of Eq. (13). Using the property that the sum $\omega_{\rm SL} + \omega_{\rm SG}$ is the total surface of the solid wall (and therefore is constant), we can derive an expression of $S_{\rm J}U_{\rm J}$ in the form

$$S_{\rm L}U_{\rm L} = U(1) - \frac{\gamma_{\rm SL}}{\gamma_{\rm SG} - \gamma_{\rm SL}}(U(S_{\rm L}) - U(1)) \tag{18}$$

$$S_{\rm G}U_{\rm G} = \frac{\gamma_{\rm SG}}{\gamma_{\rm SG} - \gamma_{\rm SL}}(U(S_{\rm L}) - U(1)) \tag{19}$$

where $U(1)$ can be set to 0 by considering the saturated state as a reference state. Since liquid is the wetting phase the fraction $\frac{\gamma_{\rm SL}}{\gamma_{\rm SG}-\gamma_{\rm SL}}$ is a positive number that we will denote by a, in the following, so that

$$S_{\rm L}U_{\rm L} = -aU(S_{\rm L})\ ; \qquad S_{\rm G}U_{\rm G} = (1+a)U(S_{\rm L}) \tag{20}$$

We have to point out that the identification (20) relies on the assumption, albeit natural, that the rhs of (17) and (13) can be identified term by term. Moreover because the surface tensions $\gamma_{\rm SJ}$ are generally not known, the coefficient a should be calibrated directly at the macroscopic scale.

3 The Equivalent Pore Pressure model

To derive this model, we will assume that along any loading path defined by a constant saturation degree, $dS_{\rm L} = 0$, and constant effective pressures, $d\pi_{\rm J} = 0$, the partial pore deformation increment, $d\varphi_{\rm J}$ is a saturation dependent fraction of the total pore deformation:

$$(d\varphi_{\rm L})_{S_{\rm L},\pi_{\rm J}} = \chi d\varphi \; ; \qquad (d\varphi_{\rm G})_{S_{\rm L},\pi_{\rm J}} = (1-\chi)d\varphi \tag{21}$$

where χ is a saturation dependent factor that varies between 1, under saturated state, and 0 under dried state. The choice $\chi = S_{\rm L}$ corresponds to the iso-deformation assumption of the two partial pore volumes: $\frac{d\varphi_{\rm L}}{\phi_0 S_{\rm L}} = \frac{d\varphi_{\rm G}}{\phi_0 S_{\rm G}}$ which has to be satisfied whatever the saturation. Accordingly, when $\chi = S_{\rm L}$, the porous network is assumed to deform homogeneously whenever no pressure is applied on the solid wall. This assumption is often used for convenient reasons [11, 27, 10].

Integration of (21) gives

$$\varphi_{\rm L} = \chi\varphi + \delta \; ; \qquad \varphi_{\rm G} = (1-\chi)\varphi - \delta \tag{22}$$

where δ is a function of $(S_{\rm L}, \pi_{\rm L}, \pi_{\rm G})$ that must vanish under saturated and dried states:

$$\delta(0, \pi_{\rm L}, \pi_{\rm G}) = \delta(1, \pi_{\rm L}, \pi_{\rm G}) = 0 \tag{23}$$

Incorporating expression (22) for $\varphi_{\rm J}$ in (5) gives

$$(dF_{\rm sol})_{S_{\rm L}} = \sigma_{ij} d\epsilon_{ij} + \pi d\varphi + \Delta d\delta \tag{24}$$

where we defined π and Δ as follows

$$\pi = \chi\pi_{\rm L} + (1-\chi)\pi_{\rm G} \tag{25}$$

$$\Delta = \pi_{\rm L} - \pi_{\rm G} \tag{26}$$

Defining the Legendre-Fenchel transform $F^*_{\rm sol} = F_{\rm sol} - \sigma_{ij}\epsilon_{ij} - \pi\varphi - \Delta\delta$ entails

$$(dF^*_{\rm sol})_{S_{\rm L}} = -\epsilon_{ij}d\sigma_{ij} - \varphi d\pi - \delta d\Delta \tag{27}$$

Since δ only depends on $(S_{\rm L}, \pi, \Delta)$, the integration of the state equation

$$\delta(S_{\rm L}, \pi, \Delta) = -\left(\frac{\partial F^*_{\rm sol}}{\partial \Delta}\right)_{S_{\rm L},\sigma_{ij},\pi} \tag{28}$$

shows that $F^*_{\rm sol}$ can be split as follows

$$F^*_{\rm sol} = F^{*1}_{\rm sol}(\sigma_{ij}, \pi, S_{\rm L}) - \int_0^\Delta \delta(S_{\rm L}, \pi, u)du \tag{29}$$

Eq. (29) suggests that $F^{*1}_{\rm sol}$ can depend on $S_{\rm L}$. Actually it cannot because of the general requirement (8)[2]. Indeed injecting equality $\Delta = 0$ in (29) and using the property (8) show that $S_{\rm L}$ is decoupled from the stresses σ_{ij} and π. Thus $F^{*1}_{\rm sol}$ only depends on (σ_{ij}, π) with

$$dF^{*1}_{\rm sol} = -\epsilon_{ij}d\sigma_{ij} - \varphi^* d\pi \tag{30}$$

where we defined

$$\varphi^* = \varphi - \int_0^\Delta \frac{\partial \delta}{\partial \pi}(S_{\rm L}, \pi, u)du \tag{31}$$

Expression (31) for φ^* can be transformed by using the Maxwell symmetry relation derived from (27)

$$\left(\frac{\partial \delta}{\partial \pi}\right)_{S_{\rm L},\Delta} = \left(\frac{\partial \varphi}{\partial \Delta}\right)_{S_{\rm L},\sigma_{ij},\pi} \tag{32}$$

[2]where we can notice that $\left(\frac{\partial F_{\rm sol}}{\partial S_{\rm L}}\right)_{\epsilon_{ij},\varphi,\delta} = \left(\frac{\partial F^*_{\rm sol}}{\partial S_{\rm L}}\right)_{\sigma_{ij},\pi,\Delta}$

We obtained

$$\varphi^* = \varphi|_{\Delta=0} \tag{33}$$

Hence φ^* turns out to be the total pore deformation that would have resulted from $\Delta = 0$, i.e. $\pi_{\mathrm{L}} = \pi_{\mathrm{G}}$, while keeping the other variables as constant.

Therefore expression $F^{*1}_{\mathrm{sol}}(\sigma_{ij}, \pi)$ which is independent of S_{L}, must have the same expression as that found under saturated state. The state equations

$$\epsilon_{ij} = -\left(\frac{\partial F^{*1}_{\mathrm{sol}}}{\partial \sigma_{ij}}\right)_{\pi} \qquad \varphi^* = -\left(\frac{\partial F^{*1}_{\mathrm{sol}}}{\partial \pi}\right)_{\sigma_{ij}} \tag{34}$$

show that π play the role of an equivalent pore pressure in the sense that it would be the pressure to apply to the liquid phase of the porous material under saturated state, to get the same strains as those obtained under unsaturated state, at the given stress state σ_{ij}. Note however that the total pore deformation that would be obtained under saturated state is φ^* and not that of the unsaturated state, φ.

Assuming that, at the microscopic scale, the solid matrix is isotropic and behaves elastically, with a compressibility constant k_s, an incremental loading defined, at constant saturation degree, by $d\sigma = -d\pi_{\mathrm{L}} = -d\pi_{\mathrm{G}} = -d\pi$, will load the solid grains by a uniform increment of pressure leading to a response given by $d\epsilon_{ii} = d\varphi/\phi_0 = d\varphi_{\mathrm{J}}/(\phi_0 S_{\mathrm{J}}) = -d\pi/k_s$. Incorporating these last equations into the general equation (22) leads to

$$\left(\frac{\partial \delta}{\partial \pi}\right)_{S_{\mathrm{L}},\Delta} = \left(\frac{\partial \varphi}{\partial \Delta}\right)_{S_{\mathrm{L}},\sigma_{ij},\pi} = \frac{\phi_0(\chi - S_{\mathrm{L}})}{k_s} \tag{35}$$

showing that δ and φ vary linearly with π and Δ, respectively. Accordingly we

obtain for φ

$$\varphi = \varphi^* + \frac{\phi_0(\chi - S_\mathrm{L})}{k_s}\Delta \tag{36}$$

showing that $\varphi^* = \varphi$ in the iso-deformation assumption, $\chi = S_\mathrm{L}$.

Furthermore the δ-Δ relationship is expected to involve the elastic shear property of the solid matrix. If we assume the linearity of such behavior we end up with

$$\delta = \frac{\phi_0(\chi - S_\mathrm{L})}{k_s}\pi + \frac{\Delta}{g(S_\mathrm{L})} \tag{37}$$

where $g(S_\mathrm{L})$ stands for an elastic modulus characterizing the elastic shear property of the solid matrix.

When the solid grains are incompressible, k_s can be set to ∞ in the previous equations. The volumetric deformation of the soil is then equal to the total pore deformation, $\epsilon_v = \varphi = \varphi^*$, and the balance (30) turns into

$$dF^{*1}_\mathrm{sol} = -\epsilon_v d(\sigma + \pi) - \epsilon_q dq \tag{38}$$

where q and ϵ_q are the deviatoric stress and strain respectively. From (38)

$$\epsilon_v = -\left(\frac{\partial F^{*1}_\mathrm{sol}}{\partial(\sigma + \pi)}\right)_q \tag{39}$$

4 Elastic behavior

Under isotropic loading, the mechanical response of saturated soils is well described by a constant compressibility coefficient, κ, in the form

$$de = -\kappa\frac{d(\sigma + p)}{\sigma + p} \tag{40}$$

where e stands for the void ratio. This coefficient has been measured on FoCa clay used for engineered barrier [14]. The compression test is shown in the figure (1) where only the BC lines are associated with the elastic response. The compressibility coefficient is $\kappa = 0.0977$.

Since energy $F^1_{\rm sol}$ has been found to be independent of the saturation degree, a comparison of (40) and (39) implies the general constitutive relationship

$$de = -\kappa \frac{d(\sigma + \pi)}{\sigma + \pi} \tag{41}$$

which should be valid for unsaturated situations and along any loading paths and with the same constant compressibility coefficient as that involved in (40). However to be able to apply the previous relation, the equivalent pore pressure, π, has to be computed from (25). This can be done from the water retention curve of the material. This curve has been measured, at 20 and 80 ^{o}C, on the same FoCa clay using the saturated salt solution technique upon imbibition path [14]. This curve is shown in the figure (2a). During these tests no stress was applied thereby, neglecting the atmospheric pressure, $\sigma = 0$. In these conditions Eq. (41) turns into

$$de = -\kappa \frac{d\pi}{\pi} \tag{42}$$

The free swelling of these samples was measured and reported in figure (2b) in terms of π calculated with the help of the measured retention curves, the gas pressure being neglecting. We assume here the iso-deformation of pores, i.e. $\chi = S_{\rm L}$. The reported measured points are found to be accurately lined up with a slope $\kappa = 0.0978$, namely very close to that found in the previous experiment.

5 Plastic Modeling

We will assume in the following that irreversibility only affects the mechanical behavior. Hysteresis of the retention curve will not be addressed here. As a consequence, in non reversible transformation, the two first laws of thermodynamics applied to the system composed of the solid matrix, gives the Clausius-Duhem inequality in the form [11]

$$\sigma_{ij} d\epsilon_{ij} + \pi_{\mathrm{L}} d\varphi_{\mathrm{L}} + \pi_{\mathrm{G}} d\varphi_{\mathrm{G}} - (dF_{\mathrm{sol}})_{S_{\mathrm{L}}} \geq 0 \tag{43}$$

where now the free energy of the solid matrix, F_{sol}, must be argumented by the elastic part of the deformation variables and by hardening variables. Following [13] this energy is split into two parts: (i) the elastic energy, W, stored in the solid matrix which is the energy recoverable by a reversible mechanical process and (ii) the locked energy, Z, which is the additional (not recoverable) part of the elastic energy locked when irreversible mechanical processes have taken place. For the sake of simplicity, the locked energy is assumed to depend on a unique hardening variable, α. Denoting with superscripts e and p the elastic and plastic part of the deformation variables, respectively, we write

$$F_{\mathrm{sol}} = W(\epsilon^e_{ij}, \varphi^e_{\mathrm{L}}, \varphi^e_{\mathrm{G}}, S_{\mathrm{L}}) + Z(S_{\mathrm{L}}, \alpha) \tag{44}$$

The state equations (7) being always valid provided that each deformation variable be replaced by its elastic part, the use of expression (44) for F_{sol} in (43) allows to write the Clausius-Duhem inequality as

$$\sigma_{ij} d\epsilon^p_{ij} + \pi_{\mathrm{L}} d\varphi^p_{\mathrm{L}} + \pi_{\mathrm{G}} d\varphi^p_{\mathrm{G}} + \beta d\alpha \geq 0 \tag{45}$$

where β is defined by

$$\beta = -\left(\frac{\partial Z}{\partial \alpha}\right)_{S_{\mathrm{L}}} \tag{46}$$

The variable β is the hardening force as energy conjugate to the hardening variable α. It will be associated to the current limit of the elastic domain defined by

$$f(\sigma, \pi_{\mathrm{L}}, \pi_{\mathrm{G}}, \beta) \leq 0 \tag{47}$$

Following the work of Coussy [13], to go further towards an effective stress, we will assume that part of the flow rule is given by

$$(d\varphi_{\mathrm{L}}^{p})_{S_{\mathrm{L}}} = \chi^{p} d\varphi^{p} \; ; \qquad (d\varphi_{\mathrm{G}}^{p})_{S_{\mathrm{L}}} = (1-\chi^{p}) d\varphi^{p} \tag{48}$$

where χ^{p} is a weighting factor ranging from 0 to 1. This factor is, a priori, different from χ which was introduced previously to describe the elastic response. Similarly to what was done for χ, we will assume that this factor depends on the saturation degree: $\chi^{p}(S_{\mathrm{L}})$. The plastic incompressibility of the solid grains is now introduced leading to $\epsilon_{v}^{p} = \varphi^{p}$. From the plastic incompressibility assumption and (48) the dissipation (45) turns into

$$\sigma^{\mathrm{B}} d\epsilon_{v}^{p} + q d\epsilon_{q}^{p} + \beta d\alpha \geq 0 \tag{49}$$

where σ^{B} is a Bishop-like stress defined by

$$\sigma^{\mathrm{B}} = \sigma + \chi^{p}\pi_{\mathrm{L}} + (1-\chi^{p})\pi_{\mathrm{G}} \tag{50}$$

According to (50) and (49) the current elastic domain can be defined by

$$f(\sigma^{\mathrm{B}}, q, \beta) \leq 0 \tag{51}$$

The flow rule is then expressed in the form

$$d\epsilon_v^p = d\lambda \left(\frac{\partial f}{\partial \sigma^{\mathrm{B}}} \right)_{q,\beta} \qquad d\epsilon_q^p = d\lambda \left(\frac{\partial f}{\partial q} \right)_{\sigma^{\mathrm{B}},\beta} \tag{52}$$

One of the simplest plastic model used for saturated clay is the Modified Cam-Clay model:

$$f_{\mathrm{Cam}}(\sigma, q, p_0) = \sigma(\sigma + p_0) + q^2/M^2 \tag{53}$$

where p_0 is the preconsolidation pressure at the saturated state which is governed by the plastic void ratio:

$$p_0 = p_r \exp\left(-\frac{e^p}{\lambda(0) - \kappa} \right) \tag{54}$$

where $\lambda(0)$ is the slope of the saturated virgin consolidation line while κ is the slope of the unloading/reloading line as introduced in section 4. A simple extension of the yield function (53) to the unsaturated state can then be formulated as

$$f = f_{\mathrm{Cam}}(\sigma^{\mathrm{B}}, q, p_0) \tag{55}$$

where the preconsolidation pressure p_0 should be extended to unsaturated situations. Following the work of Coussy [13], we set

$$p_0 = p_r h(e^p, S_{\mathrm{L}}) \tag{56}$$

where h should satisfy $h(e^p, 1) = \exp\left(-\frac{e^p}{\lambda(0)-\kappa}\right)$.

In the following we will assume that

$$h(e^p, S_{\mathrm{L}}) = h_m(e^p) h_s(S_{\mathrm{L}}) \tag{57}$$

where $h_m = \exp\left(-\frac{e^p}{\lambda(0)-\kappa}\right)$ expresses the mechanical hardening due to irreversible deformations and h_s represents a saturation-induced hardening (which

may be a function of capillary pressure through the retention curve). Note that $h_s(1) = 1$, leading to $p_0 = p_r$ at the initial (undeformed) reference saturated state.

5.1 Shear strength

Substitution of the Bishop effective stress (50) in the classical Mohr-Coulomb criterion gives

$$\tau = C - (\sigma_n + \chi^p \pi_{\mathrm{L}} + (1 - \chi^p)\pi_{\mathrm{G}}) \tan\psi \tag{58}$$

where C is the cohesion and ψ is the friction angle which is assumed constant, consistently with the Cam-Clay model: $\sin\psi = 3M/(6 + M)$. Alonso et al [3] have shown that the coefficient χ^p to consider in the shear strength of unsaturated soils is actually smaller than the saturation degree. They proposed a generic formula of the form

$$\chi^p = \left\langle \frac{S_{\mathrm{L}} - S_{\mathrm{L}}^m}{1 - S_{\mathrm{L}}^m} \right\rangle \tag{59}$$

where $< x >= (x + |x|)/2$ is the positive part operator. Alonso et al related this coefficient (called effective degree of saturation in their paper) to the freely available water filling the macroporosity of the soil. Here we will rather use this formula as a parametric form of the coefficient χ^p. From (20) and (14), a development of Eq. (58) in terms of the capillary pressure, $p_c = p_{\mathrm{G}} - p_{\mathrm{L}}$, and interface energy, U, gives

$$\tau = C - \left(\sigma_n + p_{\mathrm{G}} - \chi^p p_c - \frac{2}{3}\left(-\frac{\chi^p}{S_{\mathrm{L}}}a + \frac{1 - \chi^p}{1 - S_{\mathrm{L}}}(1 + a)\right) U\right) \tan\psi \tag{60}$$

Numerous experiments have been performed on shear strength of unsaturated soils. We present in the figure (3) those performed on Guadalix de la Sierra red clay [16] for which $C = 0$ and $\psi = 33^o$. In the same figure we have plotted the model as predicted by Eq. (60) under different hypotheses: $\chi^p = S_{\rm L}$ or χ^p given by Eq. (59). The model of Brooks and Corey [8] was used to fit the retention curve with an air entry pressure of 30 kPa and an exponent equal to 3.

5.2 Isotropic stress path at constant capillary pressure

The compression index of a normally consolidated saturated soil is defined by

$$de = -\lambda(0)\frac{d(\sigma + p)}{\sigma + p} \tag{61}$$

For an unsaturated soil at constant capillary pressure, an isotropic plastic loading on the virgin compression line, i.e $-\sigma^{\rm B} = p_0$, results in

$$de^p = -(\lambda(0) - \kappa)\frac{d\sigma^{\rm B}}{\sigma^{\rm B}} \tag{62}$$

Assuming that $\chi = \chi^p$ entails

$$de = -\lambda(0)\frac{d\sigma^{\rm B}}{\sigma^{\rm B}} \tag{63}$$

which is the simplest extension of (61) for unsaturated situations. We assume moreover, hereafter, that $\chi = \chi^p = S_{\rm L}$. We have applied this simple model to a silty clay [25, 26, 35]. The figure (4b) reports the variation of the void ratio during a compression oedometric test from 1 to 256 kPa obtained at different capillary pressures: $p_c = 0, 20, 40, 80, 160\,{\rm kPa}$. The figure (4a) shows the prediction obtained by the model. The elastic limit, given by $-\sigma = \pi + p_0$, has been

identified to $18, 22, 24, 28, 50$ kPa respectively. From these results we have identified the following parameters associated to the saturated state: $\lambda(0) = 0.037$, $\kappa = 0,004$ and $p_r = 18$ kPa. To assess the value of the interface energy, U, we use a retention curve fitted by using the model of Brooks and Corey with the parameters associated to the silty clay (air entry pressure $p_e = 10$ kPa and exponent $\alpha = 2.5$).

5.3 Imbibition drainage paths

The same silty clay as that used in the previous section was loaded at different isotropic compression stresses: $-\sigma = 8, 16, 32, 64, 128, 256$ kPa. After each compression test, the capillary pressure is increased from 0 to 160 kPa then the specimen is unloaded to its initial compression, i.e 1 kPa. The same parameters as those defined in the last section are used. Figure (5a) shows the void ratio variations during the different loading paths. The vertical lines correspond to the capillary pressure load. The corresponding evolution of the void ratio is represented in the figure (6a). The experimental results are represented in the figures (5b) and (6b).

Let's consider now a soil sample initially and normally saturated. The initial capillary pressure is equal to zero. Let's submit the specimen to increasing capillary pressure from 1 to 256 kPa. The capillary pressure is then decreased to 1 kPa. We use the following parameters $\lambda(0) = 0.19$, $\kappa = 0.031$, $p_r = 10$ kPa, and an initial stress $-\sigma_0 = 1.5\,10^{-6}$ kPa. The parameters of the retention curve (Brook and Corey) are given by $p_e = 1.8$ MPa and $\alpha = 1$. These parameters

corrrespond to a white clay used in [17]. The experiments are shown in the figure (7b) and can be compared to the predictions shown in the figure (7a). In the experiment the sample is not mechanically loaded. So we found the initial stress in order to fit one point of the curve. We found the very small value $-\sigma_0 = 1.5\,10^{-6}$ kPa. During the loading path from 1 to 1.8 kPa the capillary pressure is lower than the air entry pressure and the soil is actually saturated. The slope of the curve is the compression index. For capillary pressure greater than 1.8 kPa the sample behaves elastically because we assumed that the saturation-induced hardening $h_s(S_{\rm L}) > 1 + \frac{\chi^p p_c - p_e}{p_r h_m(e^p)}$, resulting in a strictly negative yield function: $f < 0$.

6 Conclusion

The model proposed by Coussy [13] has been extended to account for interface energies in the mechanical behaviour of unsaturated soils. This extension relies on the assumption that the interface energy depends on the partial pore deformations in addition to the saturation degree. As a consequence, effective pore pressures should be considered in the mechanical behaviour in place of the liquid and gas pressures. These effective pore pressures differ from the pore pressures by terms involving the interface energy. Following the approach developped in Coussy et al, a Bishop-like effective stress, expressed in terms of these effective pore pressures, is found to control the mechanical behaviour of unsaturated soils providing an assumption concerning the partial pore deformations, i.e the de-

formation of the partial volume occupied by the fluids. The Bishop's coefficient, χ, turns out to be a saturation dependent fraction of the partial pore deformation and the total pore deformation. Actually two Bishop-like effective stresses, associated to the elastic and plastic behaviour, can be introduced. This results in a model relying on two effective stresses which can be used to extend the elastic and plastic behaviour of saturated soils to unsaturated conditions. We propose a very simple model based on the extension of the Cam-Clay model. This model is applied to predict the response of a soil sample to compression stress at constant capillary pressure and to wetting drying paths at constant stress. These responses are compared with some experimental results reported from the literature.

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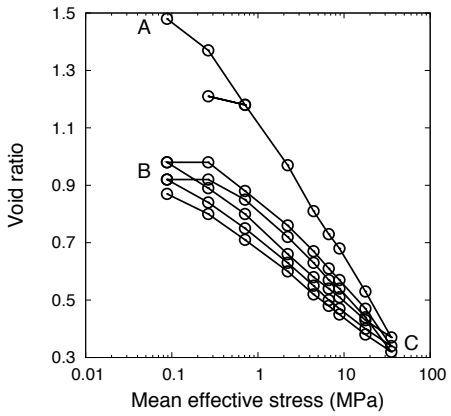
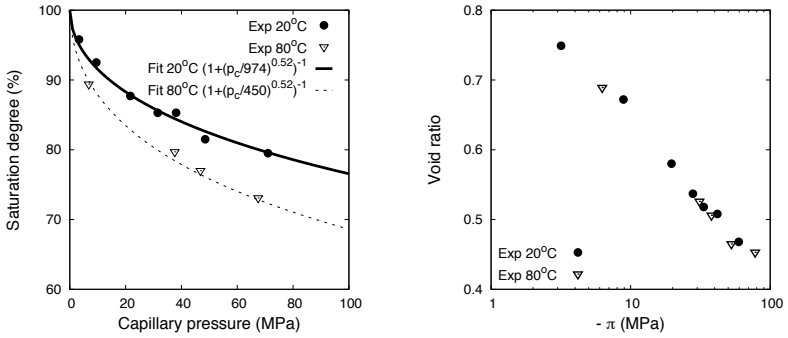
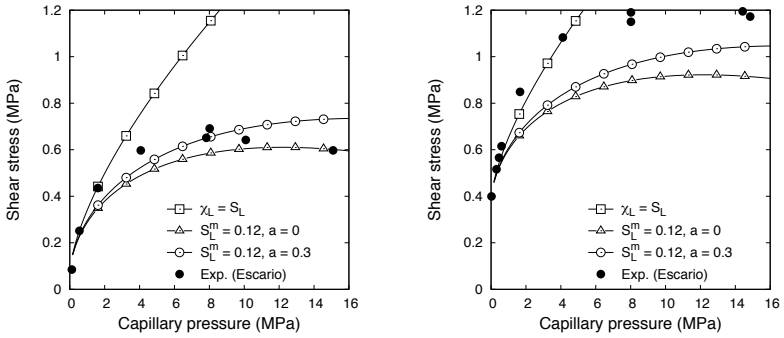


Figure 1: Isotropic compression test performed on saturated FoCa clay [14].



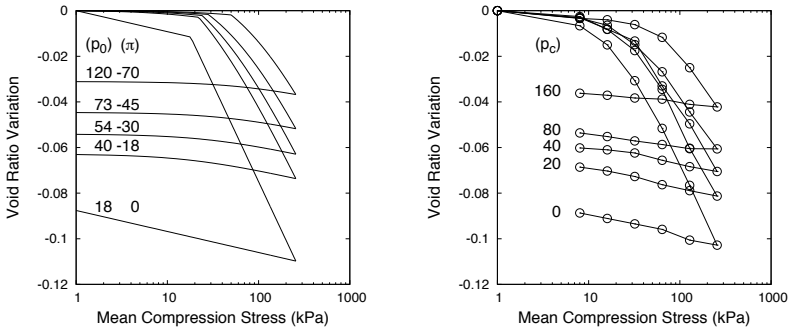
(a) (b)

Figure 2: (a) Retention curve of a FoCa clay obtained by saturated salt solution technique [14]. (b) Void ratio reported against $\log(-\pi)$ in a free swelling during imbibition test.



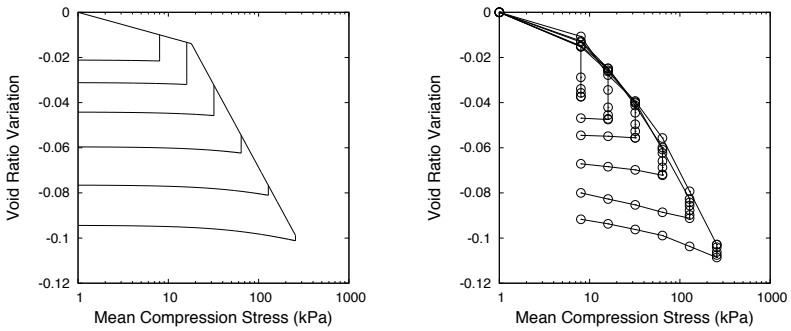
(a) (b)

Figure 3: Shear strength vs capillary pressure obtained at different normal stress: (a) 0.12 MPa (b) 0.6 MPa. The experimental results are reported from Escario et al [16].



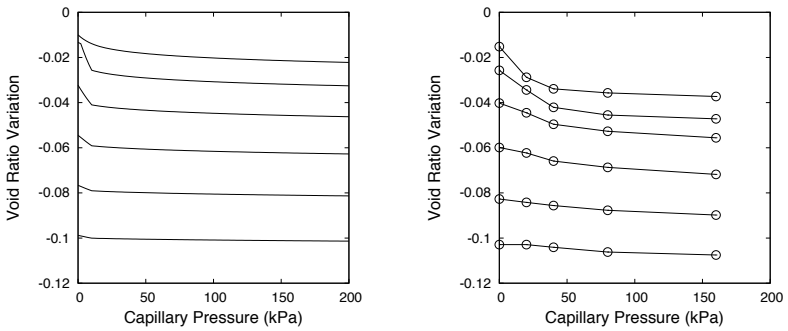
(a) (b)

Figure 4: Isotropic compression curves at constant capillary pressure obtained on a silty clay: (a) Model (b) Experiment reported from [25, 26].



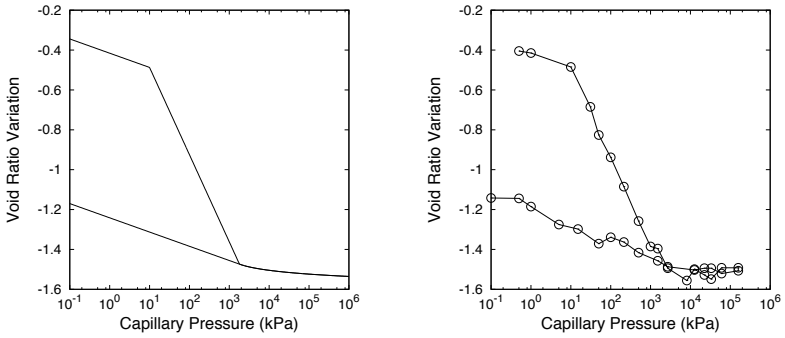
(a) (b)

Figure 5: Compression, drainage and unloading on a silty clay: (a) model (b) experiments [25, 26].



(a) (b)

Figure 6: Drainage phase on a silty clay: (a) model (b) experiments [25, 26].



(a) (b)

Figure 7: Drainage on a white clay: (a) model (b) experiments [17].