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Analyse a posteriori du schéma Chorin-Temam pour les équations de Stokes

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Abstract

We consider Chorin-Temam scheme (the simplest pressure-correction projection method) for the time-discretization of an unstationary Stokes problem in $\mathcal{D} \subset \mathbb{R}^d$ ($d = 2, 3$) given $\mu, \boldsymbol{f}$: (P) find $(\boldsymbol{u}, p)$ solution to $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$, $\boldsymbol{u}|_{\partial\mathcal{D}} = 0$ and

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mu\Delta\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \qquad \operatorname{div} \boldsymbol{u} = 0 \qquad \text{on } (0, T) \times \mathcal{D}\,. \tag{1}$$

Inspired by the analyses of the Backward Euler scheme performed by C.Bernardi and R.Verfürth, we derive a posteriori estimators for the error on $\boldsymbol{\nabla u}$ in $L^2(0, T; L^2(\mathcal{D}))$-norm. Our invesigation is supported by numerical experiments.

French version: On discrétise en temps par le schéma Chorin-Temam un problème de Stokes non-stationnaire posé dans $\mathcal{D} \subset \mathbb{R}^d$ ($d = 2, 3$) étant donnés $\mu, \boldsymbol{f}$: (P) trouver $(\boldsymbol{u}, p)$ solution de $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$, $\boldsymbol{u}|_{\partial\mathcal{D}} = 0$ et (1). En s'inspirant des analyses de C.Bernardi and R.Verfürth pour le schéma Euler rétrograde, nous construisons des estimateurs a posteriori pour l'erreur commise sur $\boldsymbol{\nabla u}$ en norme $L^2(0, T; L^2(\mathcal{D}))$. Notre étude est étayée par des expériences numériques.

Keywords: Operator splitting, Pressure corrrection, Projection method, A posteriori error estimation
French: Séparation d'opérateurs, Correction de pression, Méthode de projection, Estimation d'erreur a posteriori

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Version française abrégée

Étant donnés $\mu > 0$, $\boldsymbol{f} \in L^2(0, T; \mathrm{Q}^d)$ et $\boldsymbol{u}_0 \in \mathrm{V}$ (Q et V sont définis en (2) ci-dessous), on discrétise en temps la formulation faible (P) du problème (P) (équation (4) ci-dessous) par le schéma Chorin-Temam: soit $\boldsymbol{u}^{-1/2} = \boldsymbol{u}_0$, $p^0 = 0$, pour $n = 0 \dots N-1$, étant donnés $\Delta t^n \in (0, \Delta t]$ et $\boldsymbol{f}^{n+1} = \frac{1}{\Delta t^n}\int_{t^n}^{t^{n+1}} \boldsymbol{f}(s)ds$ (où $t_n = \sum_{k=0}^{n-1} \Delta t^k$; $t_N = T$), (P^n): on cherche $\boldsymbol{u}^{n+1/2} \in \mathrm{W}$, $p^{n+1} \in \mathring{\mathrm{Q}} \cap H^1(\mathcal{D})$ solutions de (5a–5b). La convergence a priori vers des solutions de (P) quand $\Delta t \to 0$ et son ordre sont connus [7, 8, 9] (voir Prop. 1). Mais on aimerait ici estimer *a posteriori* l'erreur de discrétisation en temps pour bien choisir les pas Δt^n en pratique, ce qui est encore un problème ouvert. Les estimateurs d'erreur a posteriori proposés dans [1, 2] pour la semi-discrétisation en temps avec le schéma Euler rétrograde ne sont pas valables ici. Et les récentes analyses [10, 11] pour la semi-discrétisation en temps avec le schéma Chorin-Temam proposent un estimateur (différent des nôtres) qui ne tient pas compte de tous les termes d'erreur.

Après avoir défini les résidus (10a–10b) d'une approximation Chorin-Temam $\boldsymbol{u}^{\Delta t}$, $p^{\Delta t}$ de la solution du problème (P) construite comme en (7), nous suivons dans ce travail la procédure générique d'analyse a posteriori des équations de Stokes instationnaires qui est présentée dans [2]. Nous testons donc l'équation (11) vérifiée par $e_u = \boldsymbol{u} - \boldsymbol{u}^{\Delta t}$, $e_p = p - p^{\Delta t}$ avec $\boldsymbol{v} = e_u - \Pi e_u \in \mathrm{V}$, $q = 0$, où Π est un opérateur de projection dans W qui conserve la divergence. De (12) dans $\mathcal{D}'(0, T)$, on tire alors l'inégalité (13) avec (8), $\operatorname{div} e_u = R_p$, (9b), $\Pi e_u = -\Pi\boldsymbol{u}^{\Delta t}$, $e_u(0) = 0$, et une intégration par parties. On obtient ensuite une borne supérieure (15) en utilisant par exemple (14a), (14b) selon [2]. D'autre part, on obtient aussi la borne inférieure (18) si en plus de (16) (obtenue facilement avec (10a), (10b) et $\operatorname{div} \boldsymbol{u} = 0$) on utilise (17). La procédure d'analyse a posteriori de [2] permet donc bien d'obtenir des bornes supérieures et inférieures complètement calculables de l'erreur $\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')}$ (voir Prop. 2 et Prop. 3). Mais d'une part, il vaut mieux utiliser $\|\operatorname{div} \partial_t \boldsymbol{u}^{\Delta t}\|_{L^2(0,t;\mathrm{Q}^d)}$ (plutôt que $\|\operatorname{div} \partial_t \boldsymbol{u}^{\Delta t}\|_{L^1(0,t;\mathrm{Q}^d)}$ si on suit strictement [2]), donc l'estimateur (22) plutôt que (20) (tiré directement de Prop. 2 et Prop. 3) si on veut une estimation robuste (c'est-à-dire de qualité indépendante des paramètres de discrétisation). D'autre part, bien que notre estimation ne soit pas totalement efficace (comme dans [10, 11], nos estimateurs ne sont pas bornés inférieurement *et* supérieurement par l'erreur), on montre néanmoins numériquement qu'elle peut être utile dans certains cas, et en particulier qu'elle est plus précise que celle proposée dans [10, 11] (plus de termes d'erreur sont pris en compte).

Pour des approximations (à $\lambda > 0$ donné) des composantes du vecteur vitesse et de la pression

$$\boldsymbol{u} = \pi \sin(\lambda t)\left(\sin(2\pi y)\sin(\pi x)^2; -\sin(2\pi x)\sin(\pi y)^2\right) \qquad p = \sin(\lambda t)\cos(\pi x)\sin(\pi y)$$

avec des éléments finis continus $\mathbb{P}_2$ et $\mathbb{P}_1$ par morceaux dans $\mathcal{D} \equiv (-1,1)\times(-1,1)$ ($d = 2$) maillé régulièrement avec des simplexes, on a calculé numériquement l'efficacité des estimateurs (20), (22) et (23) pour $t \in (0,T)$ discrétisé avec des pas de temps constants $\Delta t = T/N$ ($N \in \mathbb{N}$). En effet, notre analyse a posteriori du cas semi-discret en temps se prolonge au cas complètement discret (en décomposant les résidus discrets en composantes temporelles et spatiales comme dans [2] on obtient directement les versions discrètes en espace des estimateurs semi-discrets en temps plus des estimateurs pour l'erreur en espace), et l'erreur de discrétisation en espace est par ailleurs négligeable ici pour notre exemple numérique (comme observé dans [12] où il est utilisé pour $\lambda = 1$). Pour $\lambda = 10$, l'estimateur (22) est meilleur que (20) (qui n'est pas robuste si T est grand ou Δt petit) et (23) (dont l'efficacité diminue avec Δt car des termes d'erreur sont omis, alors qu'ils sont bien pris en compte par (22)). Toutefois, notre estimateur (22) ne représente pas toujours bien l'erreur lui non plus, même si on lui ajoute le terme $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,t;\mathrm{Q}^d)}$ de la borne supérieure (15) (a priori pas borné supérieurement par l'erreur (19)). Pour $\lambda = 1$ par exemple, l'erreur décroît avec Δt comme $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,t;\mathrm{Q}^d)}$, mais ce terme est d'un ordre de grandeur bien inférieur aux autres termes de (22) (ou (23)) donc on ne le voit que pour Δt assez petit même si on ajoute le terme $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,t;\mathrm{Q}^d)}$ à l'estimateur (22). Sans parler de l'estimation de l'erreur sur $\boldsymbol{u}$ en norme $L^\infty(0,T;L^2(\mathcal{D}))$, on n'a donc pas encore totalement résolu le problème de trouver un estimateur efficace et robuste pour l'erreur commise sur $\boldsymbol{\nabla u}$ en norme $L^2(0,T;L^2(\mathcal{D}))$ par le schéma Chorin-Temam. Il faudrait au moins ajouter des coefficients devant les termes de l'estimateur (22) plus $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,t;\mathrm{Q}^d)}$ si on veut l'utiliser en pratique. Néanmoins, nous espérons que cette étude apporte un nouvel éclairage à la question.

1. Numerical solutions to Stokes equations by Chorin-Temam pressure-correction projection method

Given a smooth bounded open set $\mathcal{D} \subset \mathbb{R}^d$ ($d = 2,3$) with boundary $\partial\mathcal{D}$ of class C^2, let us denote similarly by $(\cdot,\cdot)$ the usual L^2 inner-products for scalar and vector functions in $\mathcal{D}$ and introduce the standard functional spaces [3, 4]

$$\mathrm{Q} := L^2(\mathcal{D})\,, \quad \mathring{\mathrm{Q}} := \{q \in L^2(\mathcal{D})\,, \int_{\mathcal{D}} q = 0\}\,, \quad \mathrm{W} := [H^1_0(\mathcal{D})]^d\,, \quad \mathrm{V} := \{\boldsymbol{v} \in [H^1_0(\mathcal{D})]^d\,, \operatorname{div}\boldsymbol{v} = 0\}\,. \tag{2}$$

We consider a weak formulation of problem (P) with $\mu > 0$, $\boldsymbol{f} \in L^2(0,T;\mathrm{Q}^d)$ (given in a Bochner space), $\boldsymbol{u}_0 \in \mathrm{V}$:
(P) find $\boldsymbol{u} \in L^2(0,T;\mathrm{W})$ and $p \in L^2(0,T;\mathring{\mathrm{Q}})$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in V, and the following equation holds in $L^2(0,T)$

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{v}) + \mu(\boldsymbol{\nabla u},\boldsymbol{\nabla v}) - (p,\operatorname{div}\boldsymbol{v}) + (q,\operatorname{div}\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{v})\,,\ \forall(\boldsymbol{v},q) \in \mathrm{W}\times\mathrm{Q}\,. \tag{3}$$

It is well-known that problem (P) is well-posed [3, 4] (in particular, $\boldsymbol{u} \in C([0,T],\mathrm{V})$ so initial condition makes sense) and because of the regularity assumptions, it also holds $\partial_t\boldsymbol{u} \in L^2((0,T)\times\mathcal{D})$, $p \in L^2(0,T;H^1(\mathcal{D}))$ and in $L^2(0,T)$

$$(\partial_t\boldsymbol{u},\boldsymbol{v}) + \mu(\boldsymbol{\nabla u},\boldsymbol{\nabla v}) + (\boldsymbol{\nabla}p,\boldsymbol{v}) - (\boldsymbol{\nabla}q,\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{v})\,,\ \forall(\boldsymbol{v},q) \in \mathrm{W}\times H^1(\mathcal{D})\,. \tag{4}$$

A standard time-discretization of (4) is Chorin-Temam scheme [5, 6]: given $\boldsymbol{u}^{-1/2} = \boldsymbol{u}_0$, $p^0 = 0$, for $n = 0\ldots N-1$, given $\Delta t^n \in (0,\Delta t]$, $\boldsymbol{f}^{\Delta t} = \frac{1}{\Delta t^n}\int_{t^n}^{t^{n+1}} \boldsymbol{f}(s)ds$ ($t_n = \sum_{k=0}^{n-1}\Delta t^k; t_N = T$), ($\mathrm{P}^n$) find $\boldsymbol{u}^{n+1/2} \in \mathrm{W}$, $p^{n+1} \in \mathring{\mathrm{Q}} \cap H^1(\mathcal{D})$ solutions to

$$\left(\frac{\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^{n-1/2}}{\Delta t^n} + \boldsymbol{\nabla}p^n,\boldsymbol{v}\right) + \mu(\boldsymbol{\nabla u}^{n+1/2},\boldsymbol{\nabla v}) = (\boldsymbol{f}^{n+1},\boldsymbol{v}) \quad \forall\boldsymbol{v} \in \mathrm{V}\,, \tag{5a}$$

$$\frac{1}{\Delta t^{n+1}}(\operatorname{div}\boldsymbol{u}^{n+1/2},q) = -(\boldsymbol{\nabla}p^{n+1},\boldsymbol{\nabla}q) \quad \forall q \in \mathrm{Q}\,, \tag{5b}$$

which yields approximations whose rate of convergence to solutions of (P) is well-known *a priori* [7, 8, 9]:

Proposition 1. *The following estimate holds:*

$$\|\boldsymbol{u}^{\Delta t} - \boldsymbol{u}\|_{L^2(0,T;\mathrm{W})} + \|p^{\Delta t} - p\|_{L^2(0,T;\mathrm{Q})} = O(\Delta t^{\frac{1}{2}}) \text{ as } \Delta t \to 0 \ , \tag{6}$$

where $\boldsymbol{u}^{\Delta t}$ *and* $p^{\Delta t}$ *are defined as*

$$\boldsymbol{u}^{\Delta t}(t) = \frac{t - t_n}{\Delta t^n}\boldsymbol{u}^{n+1/2} - \frac{t - t_{n+1}}{\Delta t^n}\boldsymbol{u}^{n-1/2} , \quad p^{\Delta t}(t) = p^n . \quad \forall t \in (t_n, t_{n+1}] , \tag{7}$$

In this work, we would like to numerically evaluate *a posteriori* the time discretization error with a view to adequately choosing the time steps Δt^n of Chorin-Temam scheme in practice (under a given error tolerance), which is still an open problem. A posteriori error estimations have been proposed for the Backward-Euler scheme (including full discretizations, in time *and space*) [1, 2] but they do not straightforwardly apply here. And a posteriori analyses of Chorin-Temam scheme have indeed been carried out recently [10, 11] but they suggest an estimator (different than ours) that does not account for the whole error. The present invesigation focuses on fully computable error bounds for Chorin-Temam scheme derived from the generic a posteriori framework introduced in [2] for the unstationary Stokes equations. Although our estimator is a priori not fully efficient, it is better than other ones and useful in some cases.

Note that in the following, we denote by $a \lesssim b$ any relation $a \le Cb$ between two real numbers a, b where $C > 0$ is a numerical constant independent of the data of the problem. Moreover, we shall use standard inequalities such as

$$\|\operatorname{div} \boldsymbol{v}\|_{\mathrm{Q}} \le d^{1/2}\|\boldsymbol{\nabla}\boldsymbol{v}\|_{\mathrm{Q}^{d\times d}} , \ \forall \boldsymbol{v} \in \mathrm{W}$$

and Poincaré-Friedrichs inequality with constant $C_P(\mathcal{D}) > 0$, then also

$$\max(\|\boldsymbol{v}\|^2_{\mathrm{Q}^d}, \|\boldsymbol{\nabla}\boldsymbol{v}\|^2_{\mathrm{Q}^{d\times d}}) \le \|\boldsymbol{v}\|^2_{\mathrm{W}} \le (1 + C_P^2)\|\boldsymbol{\nabla}\boldsymbol{v}\|^2_{\mathrm{Q}^{d\times d}} , \ \forall \boldsymbol{v} \in \mathrm{W} . \tag{8}$$

In Section 2, we derive a posteriori error estimates following the procedure of [2], i.e. invoking $\Pi : \mathrm{W} \to \mathrm{W}$, a projection such that $\boldsymbol{v} - \Pi\boldsymbol{v} \in \mathrm{V}$. For all $\boldsymbol{v} \in \mathrm{W}$, $\Pi\boldsymbol{v}$ is the solution of Stokes equations: $\exists!\, q_{\boldsymbol{v}} \in \mathring{\mathrm{Q}}, \exists\Upsilon(\mathcal{D}) > 0$ such that

$$(\boldsymbol{\nabla}\Pi\boldsymbol{v}, \boldsymbol{\nabla}\boldsymbol{w}) = (q_{\boldsymbol{v}}, \operatorname{div} \boldsymbol{w}) \quad (r, \operatorname{div} \Pi\boldsymbol{v}) = (r, \operatorname{div} \boldsymbol{v}) , \ \forall(\boldsymbol{w}, r) \in \mathrm{W} \times \mathring{\mathrm{Q}} , \tag{9a}$$

$$\Upsilon\|\boldsymbol{\nabla}\Pi\boldsymbol{v}\|_{\mathrm{Q}^{d\times d}} \le \|\operatorname{div} \boldsymbol{v}\|_{\mathrm{Q}} . \tag{9b}$$

In Section 3, we numerically test our a posteriori estimator.

2. A posteriori estimation of semi-discrete errors

Let us define residuals for Chorin-Temam approximations $\boldsymbol{u}^{\Delta t}$, $p^{\Delta t}$ as in (7) of the solution to the problem (P)

$$< R_u, \boldsymbol{v} >_{\mathrm{W}',\mathrm{W}} = (\boldsymbol{f}, \boldsymbol{v}) - (\partial_t \boldsymbol{u}^{\Delta t}, \boldsymbol{v}) - (\boldsymbol{\nabla} p^{\Delta t}, \boldsymbol{v}) - \mu(\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}, \boldsymbol{\nabla}\boldsymbol{v}) \equiv (\boldsymbol{f} - \boldsymbol{f}^{\Delta t}, \boldsymbol{v}) + \mu(\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}, \boldsymbol{\nabla}\boldsymbol{v}) , \ \forall \boldsymbol{v} \in \mathrm{W} , \tag{10a}$$

$$(R_p, q) = -(\operatorname{div} \boldsymbol{u}^{\Delta t}, q) , \ \forall q \in \mathrm{Q} , \tag{10b}$$

where $\boldsymbol{f}^{\Delta t} = \boldsymbol{f}^{n+1}$, $\boldsymbol{u}^{\Delta t,+} = \boldsymbol{u}^{n+\frac{1}{2}}$ for $t \in (t_n, t_{n+1}]$. The errors $e_u = \boldsymbol{u} - \boldsymbol{u}^{\Delta t}$, $e_p = p - p^{\Delta t}$ satisfy:

$$(\partial_t e_u + \boldsymbol{\nabla} e_p, \boldsymbol{v}) + \mu(\boldsymbol{\nabla} e_u, \boldsymbol{\nabla}\boldsymbol{v}) + (\operatorname{div} e_u, q) =< R_u, \boldsymbol{v} >_{\mathrm{W}',\mathrm{W}} + (R_p, q) , \ \forall(\boldsymbol{v}, q) \in \mathrm{W} \times \mathrm{Q} \ . \tag{11}$$

Testing (11) against $\boldsymbol{v} = e_u - \Pi e_u \in \mathrm{V}$, $q = 0$, yields in $\mathcal{D}'(0, T)$ (distributional sense)

$$\frac{1}{2}\frac{d}{dt}\|e_u\|^2_{\mathrm{Q}^d} + \mu\|\boldsymbol{\nabla} e_u\|^2_{\mathrm{Q}^{d\times d}} =< R_u, e_u >_{\mathrm{W}',\mathrm{W}} - < R_u, \Pi e_u >_{\mathrm{W}',\mathrm{W}} + (\partial_t e_u, \Pi e_u) + \mu(\boldsymbol{\nabla} e_u, \boldsymbol{\nabla}\Pi e_u) . \tag{12}$$

Using Young inequality with (8), $\operatorname{div} e_u = R_p$, (9b), $\Pi e_u = -\Pi\boldsymbol{u}^{\Delta t}$, $e_u(0) = 0$, and integrating by part, one obtains

$$\|e_u\|^2_{L^\infty(0,t;\mathrm{Q}^d)} + \mu\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,t;\mathrm{Q}^{d\times d})} \lesssim \|R_u\|^2_{L^2(0,t;\mathrm{W}')} + \mu\|R_p\|^2_{L^2(0,t;\mathrm{Q})} + \int_0^t |(e_u, \partial_t\Pi e_u)| + \|(e_u, \Pi e_u)\|_{L^\infty(0,t)} . \tag{13}$$

If we follow [2], then (13) yields a computable upper-bound using the following inequalities with Young's one

$$\int_0^t \left|(e_u, \partial_t \Pi \boldsymbol{u}^{\Delta t})\right| \le \|e_u\|_{L^\infty(0,t;\mathrm{Q}^d)} \|\Pi \partial_t \boldsymbol{u}^{\Delta t}\|_{L^1(0,t;\mathrm{Q}^d)} \tag{14a}$$

$$\|(e_u, \Pi \boldsymbol{u}^{\Delta t})\|_{L^\infty(0,t)} \lesssim \|e_u\|_{L^\infty(0,t;\mathrm{Q}^d)} \|\operatorname{div} \boldsymbol{u}^{\Delta t}\|_{L^\infty(0,t;\mathrm{Q}^d)} . \tag{14b}$$

Since[1] $\|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')} \le 2\|R_u\|^2_{L^2(0,T;\mathrm{W}')} + 2\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})}$ also holds from (10a), one indeed obtains from (9b):

Proposition 2. *There exists a constant $c^+(\mathcal{D}) > 0$ such that the following computable estimations hold*

$$\begin{aligned} &\frac{1}{c^+} \max\left(\|e_u\|^2_{L^\infty(0,T;\mathrm{Q}^d)}, \|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')}, \mu\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})}\right) \\ &\le \|\boldsymbol{f} - \boldsymbol{f}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^d)} + \mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \mu\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q})} + \|\operatorname{div}\partial_t \boldsymbol{u}^{\Delta t}\|^2_{L^1(0,T;\mathrm{Q})} + \|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})} . \end{aligned} \tag{15}$$

On the other hand, from (10a), (10b) and $\operatorname{div}\boldsymbol{u} = 0$, one has

$$\mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \mu\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q})} \lesssim \|\boldsymbol{f} - \boldsymbol{f}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^d)} + \|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')} + \mu\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} , \tag{16}$$

from which one next straightforwardly obtains the counterpart of (15) if one uses, in addition to (16),

$$\|\operatorname{div}\partial_t \boldsymbol{u}^{\Delta t}\|^2_{L^1(0,T;\mathrm{Q})} \lesssim \frac{T}{\min_{n=0\ldots N-1} |\Delta t^n|^2} \|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} . \tag{17}$$

Proposition 3. *There exists a constant $c^-(\mathcal{D}) > 0$ such that the following computable lower bound holds*

$$\begin{aligned} &c^- \left(\mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \mu\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q})} + \frac{1}{N}\|\operatorname{div}\partial_t \boldsymbol{u}^{\Delta t}\|^2_{L^1(0,T;\mathrm{Q})}\right) \\ &\qquad \le \|\boldsymbol{f} - \boldsymbol{f}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^d)} + \|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')} + \left(\mu + \frac{1}{\min_{n=0\ldots N-1} |\Delta t^n|}\right)\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} . \end{aligned} \tag{18}$$

Proof of (17). We use the following inequality with $\operatorname{div}\boldsymbol{u} = 0$, noting $\frac{6}{\Delta t^n}\int_{t^n}^{t^{n+1}} \|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{\mathrm{Q}} \ge \|\operatorname{div}\boldsymbol{u}^{n+\frac{1}{2}}\|^2_{\mathrm{Q}} + \|\operatorname{div}\boldsymbol{u}^{n-\frac{1}{2}}\|^2_{\mathrm{Q}}$:

$$\|\operatorname{div}\partial_t \boldsymbol{u}^{\Delta t}\|^2_{L^1(0,T;\mathrm{Q})} \le N\sum_{n=0}^{N-1} \|\operatorname{div}(\boldsymbol{u}^{n+\frac{1}{2}} - \boldsymbol{u}^{n-\frac{1}{2}})\|^2_{\mathrm{Q}} \le 2N\sum_{n=0}^{N-1}(\|\operatorname{div}\boldsymbol{u}^{n+\frac{1}{2}}\|^2_{\mathrm{Q}} + \|\operatorname{div}\boldsymbol{u}^{n-\frac{1}{2}}\|^2_{\mathrm{Q}}) \le \sum_{n=0}^{N-1} \frac{12N}{\Delta t^n}\int_{t^n}^{t^{n+1}} \|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{\mathrm{Q}} .$$

□

So the framework introduced in [2] for an a posteriori analysis of a Backward Euler discretization of Stokes problem still applies here with Chorin-Temam scheme (it applies with any scheme provided the reconstructions $\boldsymbol{u}^{\Delta t}$, $p^{\Delta t}$ are defined using appropriate discrete variables). Though, the point is now to let not only the residuals, but also the two last terms in (13), be easily and sharply estimated (contrary to the fully discrete Backward Euler case in [2], these terms cannot be neglected here because they can be of the same order as the error). We draw the following conclusions. First, Prop. 2 and 3 suggest that the procedure of [2] should be modified here to estimate the error

$$\mu\|\boldsymbol{\nabla} e_u\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \|\partial_t e_u + \boldsymbol{\nabla} e_p\|^2_{L^2(0,T;\mathrm{W}')} \tag{19}$$

a posteriori in a more robust way than by the estimator (20) obtained straightforwardly from the estimations above:

$$\mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \mu\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,t;\mathrm{Q})} + \|\operatorname{div}\partial_t \boldsymbol{u}^{\Delta t}\|^2_{L^1(0,T;\mathrm{Q})} . \tag{20}$$

[1] Observe that the convergence of $\partial_t \boldsymbol{u}^{\Delta t} + \boldsymbol{\nabla} p^{\Delta t}$ to $\partial_t \boldsymbol{u} + \boldsymbol{\nabla} p$ in $L^2(0,T;\mathrm{W}')$ is natural here, like for Backward-Euler schemes [1].

For instance, if one replaces (14a) with the following upper bound (21), on noting (8) and (9b),

$$\int_0^t |(e_u, \partial_t \Pi e_u)| \lesssim \|\boldsymbol{\nabla} e_u\|_{L^2(0,t;\mathrm{Q}^d)} \|\operatorname{div} \partial_t \boldsymbol{u}^{\Delta t}\|_{L^2(0,t;\mathrm{Q})}\,, \tag{21}$$

then bounds similar to (15) and (18) hold but with $\|\operatorname{div} \partial_t \boldsymbol{u}^{\Delta t}\|_{L^2(0,t;\mathrm{Q})}$ instead of $\|\operatorname{div} \partial_t \boldsymbol{u}^{\Delta t}\|_{L^1(0,t;\mathrm{Q})}$ and *without invoking discretization parameters* like N and Δt^n, which suggests the a posteriori error estimator (22) more robust than (20):

$$\mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \mu\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q})} + \|\operatorname{div}\partial_t\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q})}\,. \tag{22}$$

Of course, this is not a fully efficient estimator yet, since it is a priori not bounded above *and* below by the error (19), even if one neglects the source error $\|\boldsymbol{f} - \boldsymbol{f}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^d)}$ of "high" order $O(\Delta t^2)$ – recall (6) –. It is nevertheless useful in some cases, as shown in the next section. Second, (22) sometimes improves some estimators in the literature like

$$\mu\|\boldsymbol{\nabla}\boldsymbol{u}^{\Delta t,+} - \boldsymbol{\nabla}\boldsymbol{u}^{\Delta t}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \sum_{n=0}^{N-1}\|\Delta t^{n+1}\boldsymbol{\nabla}p^{n+1} - \Delta t^n\boldsymbol{\nabla}p^n\|^2_{\mathrm{Q}} \tag{23}$$

that was proposed in [10, 11]. Clearly, for small Δt, our estimator is larger than the one proposed in [10, 11], on noting

$$\|\Delta t^{n+1}\boldsymbol{\nabla}p^{n+1} - \Delta t^n\boldsymbol{\nabla}p^n\|^2_{\mathrm{Q}} \lesssim \|\operatorname{div}\boldsymbol{u}^{n+1/2} - \operatorname{div}\boldsymbol{u}^{n-1/2}\|^2_{\mathrm{Q}} \lesssim \Delta t\left(\Delta t^n\|\operatorname{div}\partial_t\boldsymbol{u}^{\Delta t}\|^2_{\mathrm{Q}}(t)\right)$$

for $t \in (t_n, t_{n+1}]$, after using a Poincaré inequality with (5b). And, the numerical example of the following Section 3 indeed shows that (22) is a better upper-bound than (23), at least when the error is not mainly driven by $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$.

3. Numerical results

We want to bring numerical evidences that estimator (22) is sometimes i) useful and ii) better than (20) and (23). Given $\lambda > 0$, we numerically compute the efficiencies of the three estimators using discrete approximations of

$$\boldsymbol{u} = \pi\sin(\lambda t)\left(\sin(2\pi y)\sin(\pi x)^2; -\sin(2\pi x)\sin(\pi y)^2\right) \qquad p = \sin(\lambda t)\cos(\pi x)\sin(\pi y)$$

in $\mathcal{D} \equiv (-1,1)\times(-1,1)$ ($d = 2$), with $t \in (0,T)$ uniformly discretized by time steps $\Delta t = T/N$ ($N \in \mathbb{N}$) when $\mu = 1$.

We discretize in space the velocity components and the pressure with, respectively, continuous $\mathbb{P}_2$ and $\mathbb{P}_1$ Finite-Elements functions, i.e. in conforming discrete spaces $\mathrm{W}_h \subset \mathrm{W}, \mathrm{Q}_h \subset (\mathrm{Q} \cap H^1(\mathcal{D}))$ defined on regular simplicial meshes of $\mathcal{D}$. The a posteriori analysis of Section 2 still applies with right-hand side in (13) defined using now fully-discrete approximations. Then indeed, following [2], one can decompose the fully-discrete residuals in a sum of two terms, one accounting for space-discretization errors and one for time-discretization errors. The two last terms in the (new) right-hand side of (13) remain the same (they are explicitly computable). This yields estimators linked to the time discretization which are exactly the space-discrete counterparts of the terms in the bounds (15) and (18). Moreover, in our numerical example, space discretization errors prove negligible in comparison with time discretization errors (as already observed in [12] for $\lambda = 1$). We thus next show only numerical results obtained for one sufficiently fine mesh (with more than 10^5 vertices).

We compare the effectivities of (space-discrete versions of) the a posteriori error estimators (20), (22) and (23) evoked in the previous section for the (space-discrete) error $\|\boldsymbol{\nabla}e_{u_h}\|^2_{L^2(0,T;\mathrm{Q}^{d\times d})} + \|\partial_t e_{u_h} + \boldsymbol{\nabla}e_{p_h}\|^2_{L^2(0,T;\mathrm{W}')}$. One clearly sees from the numerical results obtained for $\lambda = 10, T \leq 3$ in Fig. 1 that i) (20) is not robust when Δt is too small or T too large compared with (22), and ii) (22) is better than the estimator (23) in so far as, for that specific case, it has the same decay rate than the error (19) and not a faster one like (23). Though, our estimator (22) is still not fully efficient, even when adding the term $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$ to (22), since it is not bounded above *and* below by the error. Furthermore, if we use it as such (that is as a sum of terms without coefficients), in some cases, it also fails (like (23)) at evaluating correctly the error. For instance when $\lambda = 1$, the error (19) scales like $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$ with respect to Δt, while the other terms in (22) are of higher-order in Δt. But this cannot be observed unless Δt is very small, even if we use (22) plus $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$ as an estimator[2] insofar as the magnitude of the latter term is much smaller than the

[2]Note that in fact we also added the term $\|\operatorname{div}\boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$ to (20), (22) and (23) in Fig. 1, but it is small compared to other terms, thus unseen.

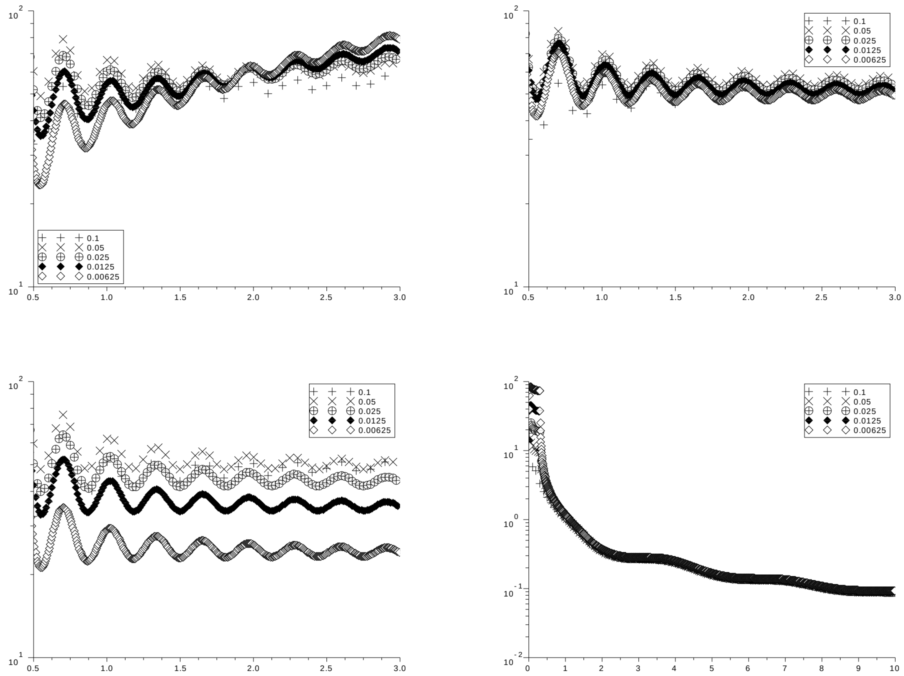

Figure 1: For $\Delta t = .1, .05, .025, .0125, .00625$, effectivities in log scale (as a function of T) of (20) – top left –, (22) – top right –, and (23) – bottom left – ($\|\operatorname{div} \boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$ included) at estimating (19) when $\lambda = 10, T \leq 3$; and $\|\operatorname{div} \boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}/$ error (19) – bottom right – when $\lambda = 1, T \leq 10$.

former (10^{-1} vs. 10^2). Then, for too large Δt, the effectivity of our estimator also decays, and the error still cannot be evaluated confidently. So, without even mentionning the error $\|e_u\|_{L^\infty(0,T;\mathrm{Q}^d)}$, the question how to estimate a posteriori error discretizations in Chorin-Temam scheme *efficiently and robustly* (in all cases) thus remains open. One should at least coefficient adequately the terms in the estimator above (22) plus $\|\operatorname{div} \boldsymbol{u}^{\Delta t}\|^2_{L^\infty(0,T;\mathrm{Q})}$. We nevertheless hope to have shed new light on the problem.

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