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Abstract

We propose a unified framework to derive thin-layer reduced models for some shallow free-surface flows driven by gravity. It applies to incompressible homogeneous fluids whose momentum evolves according to Navier-Stokes equations, with stress satisfying a rheology of viscous type (i.e. the standard Newtonian law with a constant viscosity, but also non-Newtonian laws generalized to purely viscous fluids and to viscoelastic fluids as well). For a given rheology, we derive various thin-layer reduced models for flows on a rugous topography slowly varying around an inclined plane. This is achieved thanks to a coherent simplification procedure, which is formal but based on a mathematically clear consistency requirement between scaling assumptions and the approximation errors in the differential equations. The various thin-layer reduced models are obtained depending on flow regime assumptions (either fast/inertial or slow/viscous). As far as we know, it is the first time that the various thin-layer reduced models investigated here are derived within the same mathematical framework. Furthermore, we obtain new reduced models in the case of viscoelastic non-Newtonian fluids, which extends [Bouchut & Boyaval, M3AS (23) 8, 2013].

Keywords: thin-layer reduced models ; shallow free-surface gravity flows ; Newtonian & non-Newtonian complex fluids ; viscoelastic fluids

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1. Introduction

The flow models built with Navier-Stokes equations for viscous fluids have been simplified in various ways for a long time. This has resulted in a large number of reduced models, in particular, numerous *thin-layer models* for *shallow free-surface flows* often obtained by a formal asymptotic analysis [51, 35, 40, 33].

Initially, reduced models were looked after because they were more amenable to analytical computations than full models. For instance, the Stoker and Ritter solutions to the inviscid Saint-Venant (i.e. shallow-water) equations have allowed one to model dam breaks in a simple way, with analytical formulas.

Nowadays, computer simulations often yield good approximations to full models. But good simulations of complex full models are expensive (time-consuming at least), and typically less easily interpreted from the physical viewpoint. In the case physical parameters of the model have to be explored, reduced versions of the model may thus still be preferred to full models, for instance to discriminate against various possible rheologies by comparison with experiments, see e.g. [4, 5]. Moreover, in the case where *many values* of the physical parameters have to be explored, reduced models (computationally cheaper than full models) often remain the single numerical option (even for simple toy-models, computational reductions can prove crucial, see e.g. the case of stochastic parameters in [21]). Reduced models thus remain very useful. But it is also desirable to compare them with full models, or at least one another (in the case of varying physical parameters). Now, rigorous error bounds between various (full or reduced) models are available in simple cases only, where the models remain of the same kind (see e.g. [21]). The case of free-surface shallow flow models for fluids driven by gravity is particularly striking, since various thin-layer reduced models have been proposed (see the numerous references later in this work), which are of different mathematical type, depending on various assumptions about the flow regime considered during their derivation (i.e. depending on solution properties that are not obviously connected to data, and assumed instead).

The case of free-surface flow models for perfect fluids driven by gravity (non-necessarily-shallow flows of inviscid fluids) has been treated recently: a unifying approach to irrotational water-wave models could be constructed recently [17] and extended to new reduced models with vorticity [26]. For shallow free-surface flows (of non-necessarily perfect fluids), a generic procedure has also been used recently to derive thin-layer models with various rheologies [51, 33], but it seems to hold only for the flow regimes that we later term “slow”, and it has not been used for all the cases treated in the present work (viscoelastic fluids for instance).

Our primary goal here is to establish a mathematical framework where various thin-layer reduced models obtained in various flow regimes (slow or fast), given a fixed possibly viscous rheology, can be connected one another. Moreover, we treat various rheologies (Newtonian and non-Newtonian) of viscous (also viscoelastic) fluids in the various regimes. We believe that we have thereby unified, for the first time, the derivation of many various thin-layer reduced models for shallow free-surface gravity flows, for Newtonian and non-Newtonian (viscous or viscoelastic) fluids in slow and fast regimes.

Our mathematical simplification procedure is formal. It cannot certify *rigorously* that a solution to the reduced model is a good approximation of a solution to the full model. But it is based on an intuitive *coherence* property with a clear mathematical formulation: the consistency between scaling assumptions and approximation errors in the equations. Moreover, given one rheology, we invoke successive assumptions about the flow regime until the simplification procedure delivers a closed reduced model that is coherent with the original full model. In a given flow regime, for one given rheology, our procedure is thus univoque.

The simplification procedure is inspired by [35, 40] where the viscous shallow water equations are derived from the Navier-Stokes equations for Newtonian (purely viscous) fluids (see Section 3). It aims at building a consistent approximation to a family of solutions to the initial full model, when the family of solutions defines an adequate asymptotic regime for shallow free-surface flows of incompressible viscous fluids driven by gravity on a rugous topography. Consistency is required asymptotically with respect to a nondimensional parameter $\varepsilon > 0$ parameterizing the solutions.

The asymptotic regime is defined such that only *long* free-surface waves are captured when $\varepsilon \to 0$ (i.e. only piecewise constants). The asymptotic regime in turn constrains the topographies that one can consider at the bottom of an *incompressible* flow of a *homogeneous* fluid. Precisely, in the present work, we consider only topographies defined by *slow variations around a flat plane* inclined by a constant angle θ with respect to the gravity field, thus asymptotically long waves too. Extensions with asymptotically long variations of θ seem possible [19] but are not considered here, for the sake of simplicity.

Invoking the Navier-Stokes momentum balance equations (as opposed to Euler equations), with a viscous dissipative term in the bulk along with friction boundary conditions of Navier-type on the rugous bottom of the flow, is crucial to the reduction procedure developped here (compare with [18, 20]). This modelling choice motivates the assumption (H4) : $\partial_z \boldsymbol{u}_H = O(1)$ on the shear rate, a key step to derive *coherent* reduced models (see e.g. (33) below). It is of course the responsibility of modellers to check if it makes sense for application to a real shallow flow (see Remark 1). In any case, the asymptotics $\varepsilon \to 0$ is an *idealization*. In practice, one should ask if solutions of the reduced model are close to solutions of the initial model, i.e. if they can be corrected for $\varepsilon > 0$ small and give physically-interesting answers: this justifies our *coherence* requirement.

Finally, one obtains here a synthetic view of various existing simplifications of the Navier-Stokes equations, for various rheologies and various flow regimes. Moreover, new reduced models for fluids with complex rheologies are derived.

- For viscous Newtonian fluids (modelled by the standard Navier-Stokes equations), we obtain either viscous shallow water equations in fast (inertial) flow regimes (as e.g. in [35, 40]) or lubrication equations in slow (viscous) flow regimes (as e.g. in [48, 51, 31]), see Section 4.

- For viscous non-Newtonian fluids (nonlinear power-law models), we obtain either a nonlinear version of the shallow water equations in fast flow regimes that is apparently new, or nonlinear lubrication equations in slow flow regimes (see [33] and references therein), see Section 5.

- For viscoelastic non-Newtonian fluids, we obtain either shallow water equations with additional stress terms which extends the recent work [18] in fast flow regimes, or new lubrication equations in slow flow regimes (different than those in [28, 29]), see Section 6.

A few remarks are also in order.

- The case of perfect fluids (no internal stresses) is singular. We recover it here as the inviscid limit of viscous models, provided friction is small enough at the rugous bottom boundary, and it yields the same reduced model whatever the explicit formulation of the viscous terms. One obtains the thin-layer model of Saint-Venant [52] widely used in hydraulics, also known as the nonlinear shallow-water equations. A dissipative term associated with the Navier friction boundary condition remains. It could have been derived more straightforwardly without viscosity, like in [20] for viscoelastic fluids with zero retardation time, but then also less naturally (conditions tangent to the boundary are not required for perfect fluids).

- The case of viscoplastic Non-Newtonian fluids (i.e. Bingham-type fluids) occurs as a singular limit of the nonlinear power-law models. This case is very interesting from the modelling viewpoint (some fluids are believed to possess a yield-stress, which suits well for modelling fluid-solid transitions like e.g. in avalanches). But it is also difficult from the mathematical viewpoint (the model is undetermined below the yield-stress) as well as from the physical viewpoint (the yield-stress concept is still debated [50]).
- In the case of viscoelastic fluids, we improve here the model derived in [18]. Note that the constitutive equations that we use here are simple and alike in [18]. (They are linear equations in the conformation tensor state variable, physically-consistent from the frame-invariance and from the molecular theories viewpoint.) However, here, we additionally take into account: friction at the bottom, an inclination between the constant gravity field and the main direction of the flow, surface tension, two-dimensional effects and a purely Newtonian additional viscosity (equivalently, a non-zero retardation time, from the viscoelastic rheology viewpoint).

For a physically-inclined review of thin-layer models in many possible flow regimes, we recommend [31], and the older one [48] with a focus on stability.

2. Mathematical setting of the problem

We endow the space $\mathbb{R}^3$ with a Galilean reference frame using cartesian coordinates $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$. We denote by a_x (respectively a_y, a_z) the component in direction $\boldsymbol{e}_x$ (resp. $\boldsymbol{e}_y$, $\boldsymbol{e}_z$) of a vector (that is a rank-1 tensor) $\boldsymbol{a}$, by $a_{xx}, a_{xz}, \ldots$ the components of higher-rank tensors, by $\boldsymbol{a}_H$ the vector of "horizontal" components (a_x, a_y), by $(\boldsymbol{a}_H)^\perp = (-a_y, a_x)$ an orthogonal vector, by $\nabla_H a$ the horizontal gradient $(\partial_x a, \partial_y a)$ of a smooth function $a : (x, y) \to a(x, y)$, and by $D_t a$ the material time-derivative $\partial_t a + (\boldsymbol{u} \cdot \nabla) a$. We use the Frobenius norm $|\boldsymbol{a}| = \mathrm{tr}(\boldsymbol{a}^T \boldsymbol{a})^{1/2}$ for tensors.

We consider gravity flows of incompressible homogeneous fluids, which are governed by Navier-Stokes equations (momentum balance and mass continuity)

$$\begin{cases} D_t \boldsymbol{u} = \mathrm{div}(\boldsymbol{S}) + \boldsymbol{f} & \text{in } \mathcal{D}(t)\,, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in } \mathcal{D}(t)\,, \end{cases} \tag{1}$$

on a space scale where the external force is uniform with magnitude given by the gravity constant $|\boldsymbol{f}| = g$, with the velocity field $\boldsymbol{u}$ as unknown variable, as well as Cauchy stress tensor $\boldsymbol{S} = -p\boldsymbol{I} + \boldsymbol{T}$ (where $\boldsymbol{T}$ is the deviatoric part of $\boldsymbol{S}$ when $\mathrm{tr}(\boldsymbol{T}) = 0$, and $\boldsymbol{I}$ is the identity tensor).

We consider flows where the fluid is contained for all times $t \geq 0$ within

$$\mathcal{D}(t) = \{\boldsymbol{x} = (x, y, z)\,,\ (x, y) \in \Omega_0\,, 0 < z - b(x, y) < h(t, x, y)\}\,, \tag{2}$$

between a free surface $z = b(x, y) + h(t, x, y)$ and a topography $z = b(x, y)$.

More specifically, we are interested in *shallow* flows. The two manifolds are thus assumed close to one-another in the "vertical" direction $\boldsymbol{e}_z$ whatever the "horizontal" position $(x, y) \in \Omega_0$, in comparison with a charcateristic horizontal lengthscale L. A priori, they never touch (though one usually next extends the model to cases with vacuum). To formulate this assumption, we write

$$h \sim \varepsilon$$

using a nondimensional small parameter $\varepsilon > 0$. It means that in the following, we consider *a family of solutions* to (1–2) parametrized by ε such that $h/(\varepsilon L)$ is bounded above and below uniformly in $(x, y) \in \Omega_0$, $t \geq 0$ as $\varepsilon \to 0$. In the sequel, assumption $a = O(\varepsilon)$ for a variable a simply means that the nondimensional quantity $a/(\varepsilon A)$ (where A is the natural characteristic size of a as a function of the space scale L and of a time scale T, see below) is bounded above, and may in fact decay faster than ε to zero as $\varepsilon \to 0$. We shall also use componentwise notation, e.g. $a_1, a_2 = O(\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$ for $a_1 = O(\varepsilon^{\alpha_1})$ and $a_2 = O(\varepsilon^{\alpha_2})$.

The goal of this work is to derive a closed system of equations that is satisfied approximately by a limit of the family defined above as $\varepsilon \to 0$. Hopefully, these equations define an approximate desciption of the flow that is a simpler mathematical model than (1–2) and that is useful in some physically-meaningful flow regimes at least (i.e. that is *coherent* under compatible assumptions).

We show next how to univoquely define such a "reduced" system of equations with a depth-averaging procedure inspired by [35]. We require that the new equations are obtained as an asymptotic limit of the initial full system of equations (1–2) using scaling assumptions. We also require that equations of the reduced system are consistently satisfied by solutions of the initial full system using the same scaling assumptions. (The error in the initial system approximated by solutions to the reduced one should scale similarly in ε as the approximation error of the reduced system by solutions to the full initial one.)

Quite importantly, we invoke the scaling assumption $\nabla_H(b + h) = O(\varepsilon)$ almost everywhere when $h \sim \varepsilon \to 0$, which means that the reduced models derived in this work captures only *long-wave* oscillations of the free surface.

Before turning to our reduction procedure under simplifying assumptions, let us note that the system of equations (1–2) is not closed yet at this stage. One still need to specify the rheology of the fluid (that is, invoke other equations linking $\boldsymbol{S}$ with $\boldsymbol{u}$) as well as boundary conditions. Now, we recall that it is exactly the goal of this work to derive approximations of (1–2) for *various* rheologies and flow regimes, using the same procedure, in a single framework. We thus specify later the rheology. We have in mind rheological models for:

- viscous Newtonian fluids (like water), such that the deviatoric stress tensor is a linear function of the rate-of-strain tensor $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, hence $\boldsymbol{T} = 2\eta_s\boldsymbol{D}(\boldsymbol{u})$, with η_s a *constant* kinematic viscosity,

- viscous non-Newtonian fluids, what most complex fluids are in a small range of shear rates at least, such that the mechanical behaviour is still described with a purely viscous deviatoric stress tensor, but using a nonlinear power-law $\boldsymbol{T} = 2\eta_s|\boldsymbol{D}(\boldsymbol{u})|^{n-1}\boldsymbol{D}(\boldsymbol{u})$ as viscosity (termed pseudoplastic or shear-thinning if $0 < n < 1$, dilatant or shear-thickening if $n > 1$),

- viscoelastic non-Newtonian fluids (like polymer solutions), such that $\boldsymbol{T} = 2\eta_s\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{\tau}$ invokes a non-Newtonian extra-stress $\boldsymbol{\tau}$ not necessarily deviatoric and defined through supplementary (integro-)differential equations.

Note that there is a huge amount of non-Newtonian models in the literature [6].

- Interestingly, the nonlinear power-law models for viscous non-Newtonian fluids coincide with the standard Navier-Stokes equations for viscous Newtonian fluids when $n = 1$ and with a Bingham model for viscoplastic fluids in the singular limit $n \overset{>}{\longrightarrow} 0$. Although stress is undetermined in Bingham model when $|\boldsymbol{D}(\boldsymbol{u})| = 0 \Leftrightarrow |\boldsymbol{T}| < 2\eta_s$, Bingham model can be understood as the limit of a regularized model [32] and remains the least disputed basic model for the still much debated viscoplastic non-Newtonian fluids.

- Viscoelastic fluid models have been used successfully for e.g. polymer solutions, see [15, 16]. We shall content here with simple prototypical models among numerous possibilities (see Section 6).

We believe that the various prototypical rheologies mentionned above are representative enough in order to define a unified framework for the derivation of shallow flow models.

We also believe that the following boundary conditions (BCs) shall allow us to investigate enough flow regimes (in long-wave asymptotics). Let us denote by $\boldsymbol{n} : (x, y) \in \Omega_0 \to \boldsymbol{n}(x, y)$ the unit vector of the direction normal to the bottom

$$\boldsymbol{n} = \begin{pmatrix} -\nabla_H b \\ 1 \end{pmatrix} / \sqrt{1 + |\nabla_H b|^2} \tag{3}$$

(inward the fluid) and by $(N_t, \boldsymbol{N})$ a normal at free surface (outward the fluid)

$$N_t = -\partial_t(b + h)/\sqrt{1 + |\nabla_H(b + h)|^2} \qquad \boldsymbol{N} = \begin{pmatrix} -\nabla_H(b + h) \\ 1 \end{pmatrix} / \sqrt{1 + |\nabla_H(b + h)|^2}\,. \tag{4}$$

An orthonormal frame is defined locally everywhere on the bottom using as basis in tangent planes

$$\boldsymbol{t}_1 = \begin{pmatrix} (\nabla_H b)^\perp \\ 0 \end{pmatrix} / |\nabla_H b| \qquad \boldsymbol{t}_2 = \begin{pmatrix} -\nabla_H b \\ -|\nabla_H b|^2 \end{pmatrix} / (|\nabla_H b|\sqrt{1 + |\nabla_H b|^2}) \tag{5}$$

when $|\nabla_H b| \neq 0$, otherwise $\boldsymbol{t}_1 = (0, -1, 0)^T$, $\boldsymbol{t}_2 = (-1, 0, 0)^T$. We require, at the bottom of the fluid, no penetration in the normal direction

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0\,, \quad \text{for } z = b(x,y),\ (x,y) \in \Omega_0\,, \tag{6}$$

and a Navier friction dynamic condition with coefficient k in the tangent plane

$$\boldsymbol{Sn} \wedge \boldsymbol{n} = k\boldsymbol{u} \wedge \boldsymbol{n}\,, \quad \text{for } z = b(x,y),\ (x,y) \in \Omega_0\,; \tag{7}$$

at the free surface, the usual kinematic condition $N_t + \boldsymbol{N} \cdot \boldsymbol{u} = 0$, i.e. for $t \geq 0$

$$-\partial_t(b+h) - u_x \partial_x(b+h) - u_y \partial_y(b+h) + u_z = 0\,, \text{ for } z = (b+h)(t,x,y),\ (x,y) \in \Omega_0\,, \tag{8}$$

and surface tension with coefficient γ as dynamic condition

$$\boldsymbol{SN} = \gamma\kappa\boldsymbol{N}, \quad \text{for } z = (b+h)(t,x,y),\ (x,y) \in \Omega_0, \tag{9}$$

where $\kappa(t,x,y) = -\operatorname{div} \boldsymbol{N}(t,x,y)$ is the (local) mean curvature at $z = b(x,y) + h(t,x,y)$, $(x,y) \in \Omega_0$; and finally periodic boundary conditions (for example) at the lateral boundary $\{\boldsymbol{x} = (x,y,z)\,,\ (x,y) \in \partial\Omega_0, 0 \leq z - b(x,y) \leq h(t,x,y))\}$.

Note that the question of existence (and uniqueness) of solutions to the Boundary Value Problem (BVP) above is difficult. It is precisely answered only in a few specific situations, for instance see [1, 3, 13, 54] for Newtonian viscous fluids and [45] for non-Newtonian fluids. In this work, we *assume* that specifying the BCs as above (plus initial conditions) allows one to precisely determine one solution (at least) to the bulk equations, and to scale it as a function of $\varepsilon \to 0$.

We are now ready to derive simplified equations that are verified by (univoque) approximations of the solutions to the BVP above, in the limit $\varepsilon \to 0$, *given a fixed space-time range*. To that aim, it is technically useful to rewrite the system of equations using nondimensional variables that are functions of the nondimensional *scaled* coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (t/T, x/L, y/L, z/L)$. Though, for the sake of simplicity in notations, we next abusively still write (t,x,y,z). And $\boldsymbol{u}$, p and $\boldsymbol{T}$ still denote the nondimensional variables after rescaling the dimensional ones by L/T, $(L/T)^2$ and $(L/T)^2$ respectively. We obtain nondimensional bulk equations where the nondimensional bulk force $\boldsymbol{f}$ has been rescaled by L/T^2, therefore $|\boldsymbol{f}|^{-1} = g^{-1}$ is a squared Froude number in

$$\begin{cases} D_t \boldsymbol{u} = -\nabla p + \displaystyle\sum_{i=x,y,z} (\partial_x T_{ix} + \partial_y T_{iy} + \partial_z T_{iz}) \boldsymbol{e}_i - \boldsymbol{f}\,, \\ \operatorname{div} \boldsymbol{u} = 0\,, \end{cases} \tag{10}$$

while k and γ scaled by L/T and L^3/T^2 respectively in the boundary conditions

$$(\boldsymbol{u}_H \cdot \nabla_H) b = u_z\,, \text{ for } z = b(x,y), \tag{11a}$$

$$\boldsymbol{Tn} - ((\boldsymbol{Tn}) \cdot \boldsymbol{n})\, \boldsymbol{n} - k\, (\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n}) \boldsymbol{n}) = 0\,, \quad \text{for } z = b(x,y), \tag{11b}$$

$$\partial_t h + (\boldsymbol{u}_H \cdot \nabla_H)(b+h) = u_z\,, \text{ for } z = b(x,y) + h(t,x,y), \tag{11c}$$

$$-p\boldsymbol{N} + \boldsymbol{TN} + \gamma \operatorname{div}(\boldsymbol{N}) \boldsymbol{N} = 0\,, \quad \text{for } z = b(x,y) + h(t,x,y), \tag{11d}$$

are a Navier friction coefficient and a Weber number without dimension, which respectively measure the dissipativity of the rough bottom topography, and the capillary effects due to the surface tension, with respect to the fluid inertia.

In the next sections, we follow e.g. [35, 40] and formally manipulate the equations (10–11a–11b–11c–11d) using successive assumptions about the solutions. We first consider generic Navier-Stokes equations in Section 3, then many specific rheologies in various limit regimes: Newtonian fluids in Section 4, power-law fluids in Section 5 and Oldroyd-B fluids in Section 6. In each case, after specifying scaling assumptions with respect to a parameter $\varepsilon \to 0$, a univoque closed set of reduced equations is obtained by an asymptotic procedure, such that solutions are required in turn to also satisfy asymptotically the full system of equations and therefore indeed approximate solutions to (10–11a–11b–11c–11d) in a formal way.

Finally, to conclude this mathematical introduction to our work, let us stress that we next always *assume* that the Navier-Stokes system (10–11a–11b–11c–11d) is a good model (say in comparison with experimental data), with a given rheology and *fixed* group numbers $\boldsymbol{f}, k, \gamma$. But remember that the experimental validation of (10–11a–11b–11c–11d) still raises questions by itself, see e.g. [30], including for the kind of applications that we have in mind here: i.e. *gravity-driven* flows such that $g = |\boldsymbol{f}|$ is a leading term (never constrained by scaling assumptions during our thin-layer asymptotic reduction procedure, in contrast with e.g. [10, 9, 11] where $g \to 0$). Moreover, assuming this fact, our derivation could only (formally) justify the use of reduced models for those applications at a *fixed* $0 < \epsilon \ll 1$ on a space-time range $L \times T$ to real-life shallow flows with mean depth ϵL, provided the magnitude of the fixed group numbers $\boldsymbol{f}, k, \gamma$ scale adequately with ϵ (the ϵ computed from experimental values of $T, L, \boldsymbol{f}, k, \gamma$ and mean depths should all be $\ll 1$), but not more. Now, such applications are not always the most commonly observed or useful. In particular, we discuss a bit more thoroughly physical applications with a Newtonian rheology in Remark 1, and it seems that the reduced model obtained in a fast flow regime (i.e. the Saint-Venant equations) has often been used (with success !) *beyond* our scaling assumptions. (Typically in turbulent regimes where the Navier-Stokes system with a given rheology and a fixed k at an effective boundary $z = b$ is unclear.) In fact, our work is interesting to justify the *mathematical formulation* of a reduced model, on invoking some flow regime. But it is next still desirable to check how far our reduced models can actually be interesting for application to real-life, for instance numerically. This is out of the scope of the present work, like a discussion of possible physical applications to specific real-life situations.

3. Generic reduction of free-surface gravity flows to thin-layer models

As usual for shallow free-surface gravity flows, our model reduction to thin layers in fact consists in filtering out all but the *long* free-surface waves through a formal asymptotic procedure [58, 55]. To consider incompressible fluids, we shall thus assume that the topographies asymptotically converge to a long wave too:

$$(\mathrm{H1}):\ \nabla_H b = O(\varepsilon)\,, \tag{12}$$

i.e. topographies oscillating at $O(\varepsilon)$ around a piecewise flat plane, here inclined by a *constant* angle $\Theta \in [0, \pi/2)$ with respect to a uniform gravity field

$$\boldsymbol{f} = (f_x \equiv +g\sin\Theta, f_y \equiv 0, f_z \equiv -g\cos\Theta)\,.$$

(Recall Navier-Stokes equations are Galilean invariant, and rotation is a Galilean change of frame, so Navier-Stokes equations are unchanged whatever Θ.) This already covers interesting application cases: recall extensions to more general topographies are possible but nontrivial, see e.g. [19].

Recall also equations have already been nondimensionalized to model shallow gravity flows given space and time scales L and T, such that it is natural to assume horizontal velocities $\boldsymbol{u}_H = O(1)$ bounded at least. Moreover, our asymptotic analysis shall produce model reductions on assuming that *horizontal variations remain slow* as $\varepsilon \to 0$: hence $\mathrm{div}_H\, \boldsymbol{u}_H = O(1)$ for $\boldsymbol{u}_H$ in particular.

1. We consider first the mass continuity equation $\mathrm{div}\, \boldsymbol{u} = 0$ with BC (11a)

$$u_z = \boldsymbol{u}_H|_{z=b} \cdot \nabla_H b - \int_b^z \mathrm{div}_H\, \boldsymbol{u}_H\,,$$

so (H1) and $\boldsymbol{u}_H = O(1) = \mathrm{div}_H\, \boldsymbol{u}_H$ together asymptotically imply velocity stratification $u_z = O(\varepsilon)$, and long free-surface waves by (11c), hence $\nabla_H h = O(\varepsilon)$. Note that, reciprocally, asymptotically long free-surface waves and (H1) imply the velocity stratification: long free-surface waves and velocity stratification are equivalent characterizations of the flow regimes considered herein, under (H1) and $\mathrm{div}_H\, \boldsymbol{u}_H = O(1)$. Moreover, the mass continuity equation and BCs (11a,11c) are satisfied up to errors of order $O(\varepsilon^{a,a+1,a+1})$ with $a \geq 1$, respectively, when $h, u_z, \boldsymbol{u}_H$ are replaced by approximations of order at least $O(\varepsilon^{a+1,a+1,a})$, respectively, and $\mathrm{div}_H\, \boldsymbol{u}_H, \partial_t h$ are then approximated up to $O(\varepsilon^{a,a+1})$ like $\boldsymbol{u}_H$ and h.
2. Using $u_z = O(\varepsilon)$, the momentum equation in (10) along $\boldsymbol{e}_z$ then reads

$$\partial_z p = f_z + \partial_z T_{zz} + \mathrm{div}_H\, \boldsymbol{T}_{Hz} + O(\varepsilon) \tag{13}$$

as well as for *approximations* of $\partial_z p, \partial_z T_{zz}, \mathrm{div}_H\, \boldsymbol{T}_{Hz}$ up to errors $O(\varepsilon^{1,1,1})$. Next, using $\nabla_H h = O(\varepsilon) = \nabla_H b$, one infers $\mathrm{div}\, \boldsymbol{N}(t,x,y) = -\Delta_H(b+h) + O(\varepsilon^3)$ so that BC (11d) along $\boldsymbol{e}_z$ implies (recall $\gamma \sim 1$ is a constant)

$$p|_{z=b+h} = -\gamma\Delta_H(b+h) + T_{zz} - \boldsymbol{T}_{Hz}\cdot\nabla_H(b+h) + O(\varepsilon^3)\,, \tag{14}$$

as well as for approximations of $h, p, T_{zz}, \boldsymbol{T}_{Hz}$ up to errors $O(\varepsilon^{3,3,3,2})$ (at $z = b+h$ at least) and of $\Delta_H h, \nabla_H h$ up to $O(\varepsilon^{3,3})$ like h. So, replacing $O(\varepsilon^3)$ by $O(\varepsilon^2)$ in (14) such that (14) still holds for approximations of $h, p, T_{zz}, \boldsymbol{T}_{Hz}$ up to errors $O(\varepsilon^{2,2,2,1})$ compatible with (13), it holds

$$p = f_z(z-(b+h)) - \gamma\Delta_H(b+h) + T_{zz} - \mathrm{div}_H \int_z^{b+h} \boldsymbol{T}_{Hz} + O(\varepsilon^2) \tag{15}$$

as well as for approximations of $h, p, T_{zz}, \boldsymbol{T}_{Hz}$ up to $O(\varepsilon^{2,2,2,1})$, in our flow regime (where (H1), $\operatorname{div}_H \boldsymbol{u}_H = O(1)$ imply $(u_z, \boldsymbol{u}_H) = O(\varepsilon^{1,0})$).

3. Naturally assuming $\Delta_H(b+h) = \operatorname{div}_H(\nabla_H b + \nabla_H h) = O(\varepsilon)$ (the oscillations b and h around flat planes have small amplitude with respect to the horizontal length-scale L which remain slow) and $(\boldsymbol{T}_{HH}, T_{zz}, \boldsymbol{T}_{Hz}) = O(\varepsilon^{0,0,0})$, BC (11d) along $(\boldsymbol{e}_x, \boldsymbol{e}_y)$ implies

$$\boldsymbol{T}_{Hz}|_{z=b+h} = (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H(b+h) + O(\varepsilon^2) \tag{16}$$

using (14) *with* $O(\varepsilon^3)$ *possibly replaced by* $O(\varepsilon^2)$, which implies

$$\boldsymbol{T}_{Hz}|_{z=b+h} = O(\varepsilon)\,. \tag{17}$$

And using (5), BC (11b) at $z = b$ rewrites, recalling $\nabla_H b = O(\varepsilon)$,

$$\begin{aligned}\boldsymbol{T}_{Hz} \cdot (\nabla_H b)^\perp - (\nabla_H b)^\perp \cdot \boldsymbol{T}_{HH} \nabla_H b = k\boldsymbol{u}_H \cdot (\nabla_H b)^\perp (1 + O(\varepsilon^2))\,,\\ (1 - O(\varepsilon^2))\boldsymbol{T}_{Hz} \cdot \nabla_H b - \nabla_H b \cdot \boldsymbol{T}_{HH} \nabla_H b + |\nabla_H b|^2 T_{zz}\\ = k(\boldsymbol{u}_H \cdot \nabla_H b + u_z |\nabla_H b|^2)(1 + O(\varepsilon^2))\,,\end{aligned}$$

so that if $\nabla_H b \neq 0$, one obtains with (11a)

$$\boldsymbol{T}_{Hz} = \frac{\boldsymbol{T}_{Hz} \cdot \nabla_H b}{|\nabla_H b|^2} \nabla_H b + \frac{\boldsymbol{T}_{Hz} \cdot (\nabla_H b)^\perp}{|\nabla_H b|^2} (\nabla_H b)^\perp\,,$$

and finally

$$\boldsymbol{T}_{Hz}|_{z=b} = (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H b + k\boldsymbol{u}_H + O(\varepsilon^2)\,, \tag{18}$$

which still holds if $\nabla_H b = 0$ since $\boldsymbol{T}_{Hz}|_{z=b} = k\boldsymbol{u}_H$ follows from (11b). Note that approximations (16),(18) of BCs (11d),(11b) are also satisfied by approximations of $\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}, \boldsymbol{T}_{Hz}$ up to errors $O(\varepsilon^{1,2})$ provided $k\boldsymbol{u}_H|_{z=b}$ is approximated up to an error $O(\varepsilon^2)$. But since on the other hand (15) is useful for approximations of $p - T_{zz}, \boldsymbol{T}_{Hz}$ up to $O(\varepsilon^{2,1})$ errors, notice that at this stage, a reduced model invoking approximations of $p - T_{zz}, \boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}, \boldsymbol{T}_{Hz}$ up to $O(\varepsilon^{2,1,2})$ should be looked for.

4. To proceed and construct such a closed reduced model, it remains to consider the momentum equations along $(\boldsymbol{e}_x, \boldsymbol{e}_y)$, thus

$$\partial_z \boldsymbol{T}_{Hz} \equiv D_t \boldsymbol{u}_H + \nabla_H(p - T_{zz}) - \operatorname{div}_H(\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}) - \boldsymbol{f}_H = O(1) \tag{19}$$

where all terms are naturally assumed bounded. Using (19) with BC (16) implies that shear stresses $\boldsymbol{T}_{Hz} = O(\varepsilon)$ are uniformly small, in particular $\boldsymbol{T}_{Hz}|_{z=b} = O(\varepsilon)$, which is a scaling compatible with (18) provided it holds

$$(\text{H2}) : k\boldsymbol{u}_H|_{z=b} = O(\varepsilon)\,. \tag{20}$$

In the sequel, for various rheologies, we assume scalings (i.e. flow regimes) that ensure (H2) : $k\boldsymbol{u}_H|_{z=b} = O(\varepsilon)$ and allow one to construct *coherent* reduced models (see below), compatible with the asymptotic analysis above.

To that aim, note that (19) holds for approximations of $\boldsymbol{T}_{Hz}$, $\boldsymbol{u}_H$, p, $\boldsymbol{T}_{HH}$ up to errors $O(\varepsilon^{2,1,1,1})$. Moreover, thanks to $\boldsymbol{T}_{Hz}|_{z=b} = O(\varepsilon)$, one can now replace $O(\varepsilon^2)$ by $O(\varepsilon^3)$ in (16) and (18). *This will prove crucial.*

So far we have established relations satisfied by smooth solutions to the BVP (10–11a–11b–11c–11d) as $h \sim \varepsilon \to 0$ given a topography b. And we would like to use those relations to build analytically a *reduced model* with variables

$$\begin{aligned}(h^0, \boldsymbol{u}_H^0, u_z^0, p^0 - T_{zz}^0, \boldsymbol{T}_{HH}^0 - T_{zz}^0\boldsymbol{I}, \boldsymbol{T}_{Hz}^0)\\ = (h, \boldsymbol{u}_H, u_z, p - T_{zz}, \boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}, \boldsymbol{T}_{Hz}) + O(\varepsilon^{2,1,2,2,1,2})\end{aligned} \tag{21}$$

approximating a solution to the initial BVP, given $b^0 = b + O(\varepsilon)$. Truncating

$$\begin{cases} D_t\boldsymbol{u}_H - \operatorname{div}_H(\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}) - \partial_z\boldsymbol{T}_{Hz} - \boldsymbol{f}_H = O(\varepsilon)\,, \\ \partial_z(p - T_{zz}) - f_z = O(\varepsilon)\,, \\ \operatorname{div}_H \boldsymbol{u}_H + \partial_z u_z = 0\,, \\ (\boldsymbol{u}_H \cdot \nabla_H)b - u_z|_{z=b} = O(\varepsilon)\,, \\ k\boldsymbol{u}_H + (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H b - \boldsymbol{T}_{Hz}|_{z=b} = O(\varepsilon^2)\,, \\ \partial_t h + (\boldsymbol{u}_H \cdot \nabla_H)(b+h) - u_z|_{z=b+h} = 0\,, \\ \gamma\Delta_H(b+h) + (p - T_{zz})|_{z=b+h} = O(\varepsilon^2)\,, \\ (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H(b+h) - \boldsymbol{T}_{Hz}|_{z=b+h} = O(\varepsilon^2)\,, \end{cases} \tag{22}$$

that is (10–11a–11b–11c–11d) at first-order in ε, yields the candidate

$$\begin{cases} \partial_t(h^0\boldsymbol{u}_H^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) \\ \quad = \operatorname{div}_H \displaystyle\int_{b^0}^{b^0+h^0} (\boldsymbol{T}_{HH}^0 - T_{zz}^0\boldsymbol{I}) - k\boldsymbol{u}_H^0|_{z=b^0} + h^0\boldsymbol{f}_H\,, \\ \partial_t h^0 + \operatorname{div}_H(h^0\boldsymbol{u}_H^0) = 0\,, \\ \quad p^0 - T_{zz}^0 = f_z(b^0 + h^0 - z) - \gamma\Delta_H(b^0 + h^0)\,, \\ \quad u_z^0 = (\boldsymbol{u}_H^0 \cdot \nabla_H)b^0 + \operatorname{div}_H \boldsymbol{u}_H^0(z - b^0)\,, \end{cases} \tag{23}$$

as a reduced model, accurate at first-order under assumptions (H1–H2). Of course, the system (23) is not closed yet without complementing it by equations for the rheology of the fluid. We next show that, after closing (10–11a–11b–11c–11d) in various ways, with various possible equations for the rheology of the fluid, one can obtain in a systematic fashion *closed reduced models*, on complementing (23) with equations for the rheology (simplified along the lines given above). This unifies the derivation of various thin-layer reduced models for various possible rheologies through a univoque procedure under clear assumptions.

Moreover, we would like closed reduced models that are (formally) asymptotically *coherent*, i.e. that admit for $b = b^0 + O(\varepsilon)$ a correction

$$\begin{aligned}(h^0, \boldsymbol{u}_H^0, u_z^0, p^0 - T_{zz}^0, \boldsymbol{T}_{HH}^0 - T_{zz}^0\boldsymbol{I}, \boldsymbol{T}_{Hz}^0) + O(\varepsilon^{2,1,2,2,1,2})\\ = (h, \boldsymbol{u}_H, u_z, p - T_{zz}, \boldsymbol{T}_{HH} - T_{zz}, \boldsymbol{T}_{Hz})\end{aligned} \tag{24}$$

solution to the initial BVP under the same assumptions as those used to define the reduced model. To that aim, in the sequel, we invoke other (compatible) assumptions in addition to (H1–H2).

One crucial additional assumption will be (H4) : $\partial_z \boldsymbol{u}_H = O(1)$, as already mentionned in Introduction. Because of (H4), one can consider a first-order approximation $\boldsymbol{u}_H^0 = \boldsymbol{u}_H + O(\varepsilon)$ with a flat profile $\partial_z \boldsymbol{u}_H^0 = 0$, i.e.

$$\boldsymbol{u}_H^0(t,x,y) = \boldsymbol{u}_H(t,x,y,z) + O(\varepsilon) \tag{25}$$

such that the flow asymptotically moves by "vertical slices". In particular, the depth-average of $\boldsymbol{u}_H$ satisfies (25), and it also holds for any $h^0 = h + O(\varepsilon^2)$

$$\int_b^{b+h} \boldsymbol{u}_H = h^0 \boldsymbol{u}_H^0 + O(\varepsilon^2) \text{ and } \int_b^{b+h} (\boldsymbol{u}_H \otimes \boldsymbol{u}_H) = h^0 (\boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) + O(\varepsilon^2)\,. \tag{26}$$

We discuss physical implications of (H4) at the end of the next section.

Then, in view of (22) and (26), candidates (23) can be asymptotically derived from *depth-averaged* versions of the initial model (10–11a–11b–11c–11d) [35]. Let us integrate along $z \in (b, b+h)$ the "horizontal" momentum equations (19)

$$\begin{aligned}(\boldsymbol{T}_{Hz} - (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H(b+h))|_{z=b+h}& \\ - (\boldsymbol{T}_{Hz} - (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})\nabla_H b)|_{z=b} + \int_b^{b+h} \boldsymbol{f}_H& \\ = \int_b^{b+h} D_t \boldsymbol{u}_H + \int_b^{b+h} \nabla_H (p - T_{zz}) - \operatorname{div}_H \int_b^{b+h} (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})&\end{aligned} \tag{27}$$

where Leibniz rule along with (11a–11c) allows one to rewrite the acceleration

$$\int_b^{b+h} D_t \boldsymbol{u}_H = \partial_t \int_b^{b+h} \boldsymbol{u}_H + \operatorname{div}_H \int_b^{b+h} (\boldsymbol{u}_H \otimes \boldsymbol{u}_H)\,.$$

Recalling (15), $f_z(z-(b+h)) - \gamma\Delta_H(b+h)$ approximates $p - T_{zz} = O(\varepsilon)$ up to $O(\varepsilon^2)$; (27) simplifies at first order using (16–18) (no shear stress $\boldsymbol{T}_{Hz}$ anymore)

$$\begin{aligned}\partial_t \int_b^{b+h} \boldsymbol{u}_H + \operatorname{div}_H \int_b^{b+h} (\boldsymbol{u}_H \otimes \boldsymbol{u}_H) - \operatorname{div}_H \int_b^{b+h} (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})& \\ = -k\boldsymbol{u}_H|_{z=b} + h\boldsymbol{f}_H + O(\varepsilon^2)\,.&\end{aligned} \tag{28}$$

Using (16–18) with $O(\varepsilon^3)$ instead of[1] $O(\varepsilon^2)$ note also for future reference

$$\begin{aligned}\partial_t \int_b^{b+h} \boldsymbol{u}_H + \operatorname{div}_H \int_b^{b+h} (\boldsymbol{u}_H \otimes \boldsymbol{u}_H) - \operatorname{div}_H \int_b^{b+h} (\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})& \\ = -k\boldsymbol{u}_H|_{z=b} + h\boldsymbol{f}_H + hf_z\nabla_H(b+h) + h\gamma\nabla_H\Delta_H(b+h) + O(\varepsilon^3)\,.&\end{aligned} \tag{29}$$

[1] Recall this crucial fact is due to requiring $\boldsymbol{T}_{Hz} = O(\varepsilon)$ under (H2), thus $\boldsymbol{T}_{Hz}|_{z=b} = O(\varepsilon)$.

As for the integrated continuity equation, it reads

$$\partial_t h + \operatorname{div}_H \int_b^{b+h} \boldsymbol{u}_H = 0\,. \tag{30}$$

Now, a reduced model like (23) for an approximation (24) is *coherent* with (30) and (28) as soon as it allows one to define approximations of each term in (30) and (28) up to $O(\varepsilon^2)$. In particular, recalling (26), it remains to approximate $\int_b^{b+h}(\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I})$ in (28) coherently with (22), depending on the rheology. Moreover, approximations of $k\boldsymbol{u}_H|_{z=b}$ in (28) allow one to reconstruct approximations of $\boldsymbol{T}_{Hz}$ coherently with (22), which were not obviously retrieved from (23), whereas they are necessary to show that the reduced model is fully coherent with the initial model. The idea was developped in [35] for Newtonian shallow flows with the standard Navier-Stokes equations, and it will be exactly the purpose of the next sections to exhibit similar coherent reconstructions of $\boldsymbol{T}_{Hz}$ for various rheologies (though not unique, probably).

Before proceeding, note that the depth-averaged approach above obviously invokes lower-dimensional variables that depend on the horizontal coordinates x, y but not on z the vertical one, which justifies the label "reduced model".

4. Application to Newtonian fluids

Internal stresses in Newtonian fluids are defined, after nondimensionalization, with a Reynolds number $\mathrm{Re} > 0$, by

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{HH} & \boldsymbol{T}_{Hz} \\ \boldsymbol{T}_{Hz}^T & T_{zz} \end{pmatrix} = \frac{1}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z \boldsymbol{u}_H + \nabla_H u_z \\ (\partial_z \boldsymbol{u}_H + \nabla_H u_z)^T & 2\partial_z u_z \end{pmatrix}. \tag{31}$$

Without more assumption than (H1) − (H2), one obtains, using $u_z = O(\varepsilon)$

$$\boldsymbol{T} = \frac{1}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z \boldsymbol{u}_H + O(\varepsilon) \\ (\partial_z \boldsymbol{u}_H + O(\varepsilon))^T & -2\operatorname{div}_H \boldsymbol{u}_H \end{pmatrix}.$$

Then, to derive a closed reduced model invoking coherent approximations of the stresses (such that $\boldsymbol{T}_{Hz} = O(\varepsilon)$ in particular), one needs further assumptions. Depending on the treatment of $k\boldsymbol{u}_H|_{z=b} = O(\varepsilon)$, we next show that one can in fact obtain different reduced models in the limit $\varepsilon \to 0$.

4.1. The fast flow regime

Let us specify (H2) as

$$(\mathrm{H2a}) : k \sim \varepsilon \tag{32}$$

and assume, for the scaling of $\boldsymbol{T}$ in (31) to be compatible with that in Section 3,

$$(\mathrm{H3}) : \mathrm{Re} \sim \varepsilon^{-1}\,, \qquad \text{and} \qquad (\mathrm{H4}) : \partial_z \boldsymbol{u}_H = O(1)\,. \tag{33}$$

Then one obtains $\boldsymbol{T}_{HH}, T_{zz} = O(\varepsilon^{1,1})$, and $\boldsymbol{T}_{HH} - T_{zz}\boldsymbol{I}, p - T_{zz} = O(\varepsilon^{1,1})$ using (H3). Moreover, thanks to (H4), one can look for a first-order approximation

$\boldsymbol{u}_H^0 = \boldsymbol{u}_H + O(\varepsilon)$ with a flat profile that is coherently defined with depth-average. Under (H1 − H2a − H3 − H4) an approximation $(h^0, \boldsymbol{u}_H^0) = (h, \boldsymbol{u}_H) + O(\varepsilon^{2,1})$ coherent with (30–28) up to $O(\varepsilon^2)$ can be determined as a solution to the system

$$\partial_t h^0 + \operatorname{div}_H(h^0 \boldsymbol{u}_H^0) = 0\,, \tag{34}$$

$$\partial_t(h^0 \boldsymbol{u}_H^0) + \operatorname{div}_H(h^0 \boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) + k\boldsymbol{u}_H^0 - h^0 \boldsymbol{f}_H = 0\,, \tag{35}$$

in particular; but it is not clearly coherent with the whole initial model (22).

The solutions to (34–35) allow one to construct a first-order approximation $(h^0, \boldsymbol{u}_H^0, u_z^0, p^0, \boldsymbol{T}_{HH}^0, T_{zz}^0) = (h, \boldsymbol{u}_H, u_z, p, \boldsymbol{T}_{HH}, T_{zz}) + O(\varepsilon^{2,1,2,2,2,2})$ that is obviously coherent with the continuity equation in (10), with BCs (11a–11c), and with (11d) projected along $\boldsymbol{e}_z$ when (H1 − H2a − H3 − H4) hold – recall (14) – using the reconstructions $\boldsymbol{T}_{HH}^0 = \frac{2}{\mathrm{Re}}\mathbf{D}_H(\boldsymbol{u}_H^0)$, $T_{zz}^0 = -\frac{2}{\mathrm{Re}}\operatorname{div}_H(\boldsymbol{u}_H^0)$ and

$$u_z^0 = \boldsymbol{u}_H^0 \cdot \nabla_H b + (b-z)\operatorname{div}_H \boldsymbol{u}_H^0, \quad p^0 = f_z(z-(b+h^0)) - \gamma\Delta_H(b+h^0) + T_{zz}^0\,. \tag{36}$$

But how to reconstruct approximations $\boldsymbol{T}_{Hz}^0 = \boldsymbol{T}_{Hz} + O(\varepsilon^2)$ such that (34–35) yields a fully coherent first-order approximation of the momentum balance in (10) and of the BCs (11b–11d) that have been used to derive (34–35) asymptotically under (H1 − H2a − H3 − H4) ? For instance choosing $\frac{1}{\mathrm{Re}}\nabla_H u_z^0 = O(\varepsilon^2)$ for $\boldsymbol{T}_{Hz}^0$ does not imply $\boldsymbol{T}_{Hz}^0|_{z=b} = k\boldsymbol{u}_H^0 + O(\varepsilon^2)$ a priori.

To that aim, we follow [35] and construct a first-order approximation $\tilde{\boldsymbol{u}}_H^0 = \boldsymbol{u}_H^0 + \boldsymbol{u}_H^1$ with a correction $\boldsymbol{u}_H^1 = O(\varepsilon)$ such that it holds for $\tilde{\boldsymbol{u}}_H^0 = \boldsymbol{u}_H + O(\varepsilon)$

$$\int_b^{b+h} \boldsymbol{u}_H = h^0 \tilde{\boldsymbol{u}}_H^0 + O(\varepsilon^3) \text{ and } \int_b^{b+h} (\boldsymbol{u}_H \otimes \boldsymbol{u}_H) = h^0(\tilde{\boldsymbol{u}}_H^0 \otimes \tilde{\boldsymbol{u}}_H^0) + O(\varepsilon^3) \tag{37}$$

given $h = h^0 + O(\varepsilon^2)$. The first-order approximation $\tilde{\boldsymbol{u}}_H^0$ remains coherent with (30–28) provided $\boldsymbol{u}_H^0$ still satisfies (34–35) up to $O(\varepsilon^2)$ so that it holds

$$\partial_t \boldsymbol{u}_H^0 + (\boldsymbol{u}_H^0 \cdot \nabla_H)\boldsymbol{u}_H^0 + k\boldsymbol{u}_H^0/h^0 = \boldsymbol{f}_H + O(\varepsilon)\,. \tag{38}$$

Then $\tilde{\boldsymbol{u}}_H^0$ will also satisfy the horizontal projection of the momentum equation (10) up to $O(\varepsilon)$ provided $\partial_z \boldsymbol{T}_{Hz}^0 = -k\boldsymbol{u}_H^0/h^0 + O(\varepsilon)$ holds for some approximation $\boldsymbol{T}_{Hz}^0$. Now, the ansatz $\tilde{\boldsymbol{u}}_H^0 = \boldsymbol{u}_H^0 + \boldsymbol{u}_H^1$ for a *coherent* first-order approximation of $\boldsymbol{u}_H$ in (19) – where $\boldsymbol{u}_H^1 = O(\varepsilon)$ – requires $\partial_z \boldsymbol{T}_{Hz} = \frac{1}{\mathrm{Re}}\partial_{zz}^2 \boldsymbol{u}_H^1 + O(\varepsilon)$, so that, recalling (38) for $\boldsymbol{u}_H^0$ solution to (34–35), we shall require

$$\partial_z \boldsymbol{T}_{Hz}^0 \equiv \frac{1}{\mathrm{Re}}\partial_{zz}^2 \boldsymbol{u}_H^1 = -\frac{k}{h^0}\boldsymbol{u}_H^0 + O(\varepsilon)\,, \tag{39}$$

thus in fact, with (H3) i.e. $\boldsymbol{T} = O(\varepsilon)$ implying $\boldsymbol{T}_{Hz}|_{z=b+h} = O(\varepsilon^2)$ in (16),

$$\boldsymbol{T}_{Hz}^0 \equiv \frac{1}{\mathrm{Re}}\partial_z \boldsymbol{u}_H^1 = k\boldsymbol{u}_H^0 \frac{b^0 + h^0 - z}{h^0} + O(\varepsilon^2)\,. \tag{40}$$

To ensure that one can build a coherent reduced model with such a corrected first-order approximation $\tilde{\boldsymbol{u}}_H^0 = \boldsymbol{u}_H^0 + \boldsymbol{u}_H^1$, we follow [35] and finally require

$$\boldsymbol{u}_H^1 = \frac{\mathrm{Re}k}{2h^0}\left((b^0 + 3h^0/2 - z)(z - b^0 - h^0/2) + |h^0|^2/12\right)\boldsymbol{u}_H^0\,. \tag{41}$$

Then, on account of $\boldsymbol{u}_H^1 = O(\varepsilon)$, all the scalings derived above for $\boldsymbol{u}_H^0$ without correction remain true for $\tilde{\boldsymbol{u}}_H^0$. Additionally, a reduced model for such a $\tilde{\boldsymbol{u}}_H^0$ is now fully coherent with all the equations of the initial model in (22). It remains to compute $\boldsymbol{u}_H^0$ such that it coherently accounts for the correction. Now, recall (37), so that the *second-order* approximation of the depth-averaged equation obtained after truncating (29) should be used coherently with $\int_b^{b+h} \boldsymbol{u}_H^1 = O(\varepsilon^3)$ and $\tilde{\boldsymbol{u}}_H^0|_{z=b} = \boldsymbol{u}_H^0(1 - h\mathrm{Re}k/3)$ as a closed system of equations for $(h^0, \boldsymbol{u}_H^0)$,

$$\partial_t h^0 + \mathrm{div}_H(h^0 \boldsymbol{u}_H^0) = 0\,, \tag{42}$$

$$\begin{aligned}&\partial_t(h^0\boldsymbol{u}_H^0) + \mathrm{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) - \left(h^0\boldsymbol{f}_H + f_z h^0 \nabla_H(b+h^0)\right) + k\boldsymbol{u}_H^0(1 - \frac{h^0\mathrm{Re}k}{3})\\ &= \gamma h^0 \nabla_H \Delta_H(b+h^0) + \frac{2}{\mathrm{Re}}\,\mathrm{div}_H\left(h^0\left(\mathbf{D}_H(\boldsymbol{u}_H^0) + \mathrm{div}_H\,\boldsymbol{u}_H^0 \boldsymbol{I}\right)\right)\,,\end{aligned} \tag{43}$$

to define a coherent reduced model when (H1 – H2a – H3 – H4) hold.

Let us stress that the new reduced model (42–43) defines only a *first-order* approximation of the initial model, coherent with all equations in (22) at first-order (up to $O(\varepsilon)$ in the bulk and $O(\varepsilon^2)$ at the boundaries), although it approximates the depth-averaged momentum equation (29) at second-order.

It is noteworthy that unlike (34–35) the system (42–43) has exactly the structure of the Saint-Venant equations [52] which have been used for a long time in numerous physical applications invoking shallow free-surface flows. The system (42–43) has indeed retained most interesting features of the initial system (except vertical acceleration), which is likely to be useful for physical applications with stratified velocities and a pressure close to the hydrostatic equilibrium such as encountered in geophysics. It also justifies the energy as a mathematical entropy usually associated with the hyperbolic system (42–43) in the inviscid limit $\mathrm{Re}^{-1} \to 0$. (The inviscid limit has been used more often in applications, for instance in hydraulics where water is often considered as a perfect fluid [58, 55], but it cannot justify by itself the choice of a mathematical entropy, while the latter is crucial to the computation of shocks ; see also [35].)

Note however that our model reduction procedure does not justify all the physical applications of the reduced model (42–43), which is beyond the scope of this work (see also Remark 1). To that aim, one still needs investigating other asymptotic justifications, and possibly in turn improve the reduced model for real-life applications. For the sake of illustration, let us simply mention that the inviscid system (42–43) without friction can also be obtained as a first-order reduced model by an asymptotic analysis of the *irrotational Euler equations* in the shallow water regime [17, 26]. But clearly, (42–43) does not only lack a description of surface waves dispersion, as it is well recognized for applications to water waves in the ocean [26], it also lacks a correct account for the flow vorticity $\mathrm{curl}\,\boldsymbol{u} = (\partial_z \boldsymbol{u}_H, \omega = \nabla_H \wedge \boldsymbol{u}_H) + O(\varepsilon)$. For instance, coherence with

$$\partial_t \omega + (\boldsymbol{u}_H \cdot \nabla_H)\omega = \omega \partial_z u_z + O(\varepsilon) \Leftrightarrow \partial_t \omega + \mathrm{div}_H(\omega \boldsymbol{u}_H) = O(\varepsilon)$$

the vertical vorticity equation, on noting $\partial_z \omega = \nabla_H \wedge \partial_z \boldsymbol{u}_H = O(1)$ under (H4), is satisfied by the first-order approximation $\omega^0 := \nabla_H \wedge \tilde{\boldsymbol{u}}_H^0$ of ω, because the

$O(\varepsilon)$ friction term can be neglected. But it is not clear that $\partial_z \tilde{\boldsymbol{u}}_H^0$ coherently approximates the horizontal components of the vorticity unless one assumes something about $\partial_{zz}^2 \tilde{\boldsymbol{u}}_H$. Future works may want to better take into account the vorticity (asymptotically) similarly to [26] for viscous cases. This may in turn help at better accounting for the stronger energy dissipation actually needed in practice to match some physical applications (see Remark 1).

Note last that scaling assumptions in this section do not restrict the velocity range, as opposed to the next section, hence the label *fast* for the present flow regime. We could also have termed it *inertial* because of the leading term in the momentum equation that balances gravity, as opposed to *viscous* in the next section (in our *slow* flows, gravity is balanced by dominant viscous effects).

Remark 1 (Hydraulic applications of the Newtonian reduced model). *For applications, recall first that we have initially assumed the initial Navier-Stokes model (10–11a–11b–11c–11d) good for physics (i.e. the flow model can actually be fitted to real-life data on adjusting the non-dimensional numbers k, γ and the rheology). This is a limitation: it has rarely been checked thoroughly, and is still under investigation as concerns gravity-driven free-surface flows [30]. Note that successful applications of the reduced model (42–43) to real-life situations would also straightforwardly reveal that the initial model is good... but only in regimes compatible with the scaling assumptions used for coherence !*

For application to water flows in contact with air, at fixed pressure and temperature in a laboratory, it seems natural to use the Navier-Stokes free-surface model (10–11a–11b–11c–11d) with (i) a fixed Froude number g so that one gets the relationship $L = (G/g)T^2$ in a uniform gravity field $G = 10[ms^{-2}]$, and (ii) a Newtonian rheology where the viscosity η_s takes values of magnitude $10^{-6}[m^2s^{-1}]$ (coinciding with measures in laminar flows and with the definition from molecular models close to thermodynamical equilibrium). It is also natural to require (iii) vanishing surface tension when (H3) : $Re^{-1} = O(\varepsilon)$ holds, insofar as the nondimensional Weber number reads $\gamma = (\gamma_{wa}/G)/L^2 = (\gamma_{wa}/G)(\sqrt{G}/\eta_s Re)^{4/3}$, $\gamma_{wa} = 10^{-5}[m^3s^{-2}]$ being the tension on a water interface with air. Then, the model (10–11a–11b–11c–11d) has actually been used to reproduce numerically fast waves in water (i.e. a lock-exchange with air in a rectangular channel) such as in the celebrated experiment of Martin and Moyce [42], e.g. with $g = 10$ and $Re \geq 10^5$ [41], though admittedly for large k and large aspect ratio. (Friction due to the boundary conditions is not small in the experiment of Martin and Moyce, as well as the aspect ratio of the flow depth with respect to the length scale $L \approx 1[m]$.) Moreover, numerical simulations of the same Navier-Stokes equations in a similar setting where k and the aspect ratio actually go to zero have next confirmed the validity of the reduced model (42–43) in the fast flow regime. In particular, assuming $k = O(\varepsilon^2)$ (stronger than (7)) or simply the pure-slip limit case $k = 0$, the stronger motion-by-slice $\partial_z \boldsymbol{u}_H = O(\varepsilon)$ holds while the solution $\boldsymbol{u}_H^0$ to (42–43) with a vanishing friction straightforwardly defines a first-order coherent approximations. And it is well-known that such a solution is well-approximated by the vanishing-viscosity limit $Re^{-1} \to 0$ of the system (42–43), which has also often been used as a reduced *model for long waves [58, 55] instead of the viscous shallow water equations. In particular, the Ritter (analytical) solution to the inviscid model (42–43) compares well [25] with numerical simulations of the Navier-Stokes equations in the long-wave asymptotic regime. This shows physical interest for the reduced model.*

However, the asymptotic comparisons above, in regimes with little energy dissipation, do not justify the interest of the reduced model for current application to real-life long-wave settings, such as dam-breaks and fast floods in rivers.

There has been numerous applications of the viscous Saint-Venant equations (42–43) to flows in rivers [36], at Froude numbers $g \approx 1$ *where one gets the relationship* $L = (G/g)T^2$ *in a uniform gravity field* $G = 10[ms^{-2}]$ *and neglects the surface tension effects. Then, one has typically used a newtonian rheology with viscosity* $\eta_s \approx 10^{-4}[m^2s^{-1}]$ *and friction values of magnitude* $k_d \approx 1[ms^{-1}$ *[8]. This is compatible with small nondimensional numbers* $Re^{-1}, k = O(\varepsilon^{1,1})$ *– and therefore with our scaling assumptions* (H1 − H2a − H3 − H4) *– provided the initial Navier-Stokes model has been nondimensionalized with space-scale* $L = (k_d/k)^2(g/G) = (\eta_s Re/G)^{2/3}$*, then for small water depths* $h \sim \varepsilon L$ *of magnitude* $10^{-2}[m]$ *only. As a consequence, our asymptotic procedure cannot justify the success of numerical computations with the reduced model (42–43) for river situations where the Navier-Stokes model a more detailed one.*

In view of the large values used for η_s *in practice for rivers (in comparison with the molecular value) and the fact that* k *is a calibration parameter, the problem with our reduced model (42–43) is that too little energy seems actually dissipated by the flow model in comparison with actual river flows.*

One explanation for the interest of Saint-Venant models like (42–43) in hydraulics is that they can also be derived by a reduction procedure initially starting with another detailed model than (10–11a–11b–11c–11d), typically one allowing for more energy dissipation like a model with turbulence effects [56].

It is also possible that the standard Saint-Venant model (42–43) is not that adequate for river hydraulics. Indeed, Saint-Venant depth-averaged models only take into account motion by slices, and neglect the boundary condition at the bottom. Now, since a clear definition of a fictitious boundary $z = b$ *where effective friction boundary conditions of Navier type apply remains a challenge (see also Remark 2), one could alternatively try to enforce a no-slip boundary condition* ($k \to \infty$) *at* $z = b$*, similarly to what occurs at the roughness scale on a river bed. To that aim, one needs another reduction procedure, leading to another two-equations reduced model like Saint-Venant. We are only aware of the one used in e.g. [24], which yields a model coinciding at first-order with the lubrication model that we obtain in the next section, and which is therefore coherent with slow flows in a viscous regime (in fact, the first-order* (50) *has usually been applied to thin films, with capillary effects [48], at smaller scales than rivers). Unfortunately, the latter alternative is not satisfactory for river flows either. In particular, it is coherent with value* $6/5 = 1.2$ *for the Boussinesq coefficient, which is higher than what is usually observed experimentally [27].*

4.2. The slow flow regime

Instead of assuming (H2a) to achieve (H2), let us now assume $k \sim 1$ and

$$(\text{H2b}) : \boldsymbol{u}_H|_{z=b} = O(\varepsilon)\,. \tag{44}$$

Moreover, to satisfy $\boldsymbol{\tau}_{Hz} = O(\varepsilon)$ (recall Section 3), we require (H3) : $\text{Re} \sim \varepsilon^{-1}$ and (H4) : $\partial_z \boldsymbol{u}_H = O(1)$. Then, on using (H2b) and (H4), it holds $\boldsymbol{u}_H = O(\varepsilon)$, which is stronger than (H2b). It also holds $u_z = O(\varepsilon^2)$ by (11a) and the continuity equation. So the viscous terms balance forces in momentum equations

$$\frac{1}{\text{Re}} \partial^2_{zz} \boldsymbol{u}_H = \boldsymbol{f}_H + O(\varepsilon) \tag{45}$$

and with the BC (17) $\partial_z \boldsymbol{u}_H|_{z=b+h} = O(\varepsilon)$, we obtain in that flow regime

$$\frac{1}{\text{Re}} \partial_z \boldsymbol{u}_H = \boldsymbol{f}_H (z - (b+h)) + O(\varepsilon^2). \tag{46}$$

A coherent approximation should also satisfy the boundary condition (18)

$$\frac{1}{\text{Re}} \partial_z \boldsymbol{u}_H|_{z=b} = k \boldsymbol{u}_H + O(\varepsilon^3)\,. \tag{47}$$

So one can use, for approximation of $\boldsymbol{u}_H$, its truncation at first-order

$$\boldsymbol{u}_H = \boldsymbol{f}_H \left(\text{Re} \left((z - (b+h))^2/2 - h^2/2 \right) - h/k \right) + O(\varepsilon^2)\,, \tag{48}$$

and finally obtain an autonomous equation for $h^0 = h + O(\varepsilon^2)$ using

$$\int_b^{b+h} \boldsymbol{f}_H \left(\text{Re} \left((z - (b+h))^2/2 - h^2/2 \right) - h/k \right) = -\boldsymbol{f}_H \left(\text{Re} \frac{h^3}{3} + \frac{h^2}{k} \right) \tag{49}$$

as an approximation of $\int_b^{b+h} \boldsymbol{u}_H$ up to order $O(\varepsilon^3)$. The solution to

$$\partial_t h^0 - \text{div}_H \left(\boldsymbol{f}_H \left(\text{Re} \frac{|h^0|^3}{3} + \frac{|h^0|^2}{k} \right) \right) = 0 \tag{50}$$

allows one to define a coherent approximation of large-time steady flow solutions to the initial BVP, such that $\partial_t h^0 = \partial_t h + O(\varepsilon^3)$ and $\partial_t \boldsymbol{u}_H = O(\varepsilon)$, as long as $(\text{H1} - \text{H2b} - \text{H3} - \text{H4})$ hold, using a first-order approximation $\boldsymbol{u}_H^0$ reconstructed from h^0 with (48) and u_z^0, p^0 reconstructed like in the previous section.

Note that (50) is exactly (2.28) in [48], where one also comments on the fact that this reduced model is strongly reminiscent of Reynolds lubrication equation [12] except that here one has a free-surface condition, so the pressure is known to be hydrostatic, while the boundary $z = h$ is unknown. This is the first time that a lubrication equation is derived in the same framework as the shallow water equations of the previous section, to our knowledge.

Lubrication equations obtained in a viscous slow-flow regime have a number of applications, similarly to the shallow water equations obtained in the inertial regime, though in different situations, see e.g. [48]. A regime where viscous forces dominate the inertial terms to balance gravity better suits slow flows on long times in small domains, where boundary effects are important and k can be chosen as large as necessary to approximate the no-slip boundary condition obtained in the $k \to \infty$ limit, as opposed to an inertial regime that better suits fast flows on short times in large domains, where boundary effects can be modelled by an effective friction condition on a fictitious boundary. But of course this description is only phenomenological and not quantitatively useful. Real flows are the result of particular initial and boundary conditions, and the question how to adequately choose a priori one of the two kinds of reduced models (or none !) may be difficult.

For application to thin films in industrial processes, one has often been interested in enriched versions of (50) with larger surface tension effects [44]. And more generally, to include small-times effects in the lubrication models of surfaces waves, one has tried to pursue the asymptotic procedure after the first-order approximation (48). The resulting models typically involve higher-order derivatives of h (which is a difficulty for numerical simulations) and the coherence of these approximations is not obvious anymore. Remarkably though, one can *rigorously* prove that the depth-averaged solutions to Navier-Stokes equations in a specific "slow flow" regime (a no-slip case $k \to \infty$ with large capillary effects $\gamma \to \infty$, see [24]) asymptotically satisfy a two-equations system like the Saint-Venant equations (containing high-order derivatives, and the $k \to \infty$ limit of (49) at first order). We refer to [24, 39, 33] for details.

Finally, one may also want to take profit of the fact that two reduced models have been identified, that are two coherent asymptotics of the same Navier-Stokes equations, and use each of them in two distinct flow subdomains. But to that aim, one still needs to precise coherent interface conditions in between the subdomains, and this remains a challenge (see Remark 2).

Remark 2 (Matching the two regimes with interface conditions).
For application to real free-surface flows, one may want to define a two-layer model by superimposing a slow viscous lower-layer and a fast inertial upper-layer, so as to better take into account i) the no-slip boundary condition at a true rough bottom and ii) the notion of effective friction at a fictitious boundary (the interface). To that aim, the problem is how to define an interface $z = b+Y$ $(0 \le Y \le h)$ *between the two layers.*

For instance, assuming there is no mass-exchange between the two incompressible fluid layers, the interface would define a fictitious free-surface for the lower-layer (the continuity equation would still yield an autonomous evolution equation for Y *where the viscous regime holds), and furthermore a fictitious topography for upper-layer. But to match the two flow regimes defined above, a first obvious difficulty is: such a construction would necessarily require the horizontal velocities* $\boldsymbol{u}_H$ *to be discontinuous at the interface (at least in the limit* $\varepsilon \to 0$*, which implies that* $\partial_z \boldsymbol{u}_H$ *hence also* $\boldsymbol{T}_{Hz}$ *is unbounded close to the interface). Moreover, there seems to be no straightforward procedure to define an interface* Y *and a coefficient* k*: one needs to introduce additional assumptions* *to that aim.*

One strategy may be to find an intermediate transition layer with depth $\eta \in (0, h - Y)$*,* $\eta = o(\varepsilon)$*, a velocity field* U *solution to the momentum equations in* $z \in (b+Y, b+Y+\eta)$*, and a friction coefficient* $k \sim \varepsilon$ *(possibly also a tension* γ*) such that when* $\varepsilon \to 0$*:*

- *the limit of* $U|_{z=b+Y}$ *is a good approximation (at* $O(\varepsilon^2)$*) of the limit of* $\boldsymbol{u}|_{z=(b+Y)^-}$ *which is given by the velocity solution to the viscous regime in* $z \in (b, b+Y)$*,*
- *the limit of* $U|_{z=b+Y+\eta}$ *is a good approximation (at* $O(\varepsilon)$*) of the limit of* $\partial_z \boldsymbol{u}, \boldsymbol{u}|_{z=(b+Y+\eta)^+}$*, which is given by the solution to the inertial regime in* $z \in (b+Y, b+h)$ *with a friction coefficient* k*,*
- $\partial_z U/(kReU)|_{z=b+Y+\eta}$ *has a limit so that Navier friction law holds at* $z = b+Y+\eta$*,*
- *normal stresses are continuous at* $z = b+Y$ *(and one may want to define a tension coefficient* γ *at* $z = (b+Y)^+$ *to formulate this).*

It is reminiscent of the boundary-layer theory for turbulent flows, but has not been achieved in the context of free-surface flows yet to our knowledge.

5. Application to purely-viscous non-Newtonian fluids

Purely viscous non-Newtonian fluids can be described by a power-law model

$$\begin{aligned}\boldsymbol{T} &= \frac{|\mathbf{D}(\boldsymbol{u})|^{n-1}}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z \boldsymbol{u}_H + \nabla_H u_z \\ (\partial_z \boldsymbol{u}_H + \nabla_H u_z)^T & 2\partial_z u_z \end{pmatrix} \\ &= \frac{|\mathbf{D}(\boldsymbol{u})|^{n-1}}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z \boldsymbol{u}_H + O(\varepsilon) \\ (\partial_z \boldsymbol{u}_H + O(\varepsilon))^T & -2\,\mathrm{div}_H\, \boldsymbol{u}_H \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_{HH} & \boldsymbol{T}_{Hz} \\ \boldsymbol{T}_{Hz}^T & T_{zz} \end{pmatrix} \end{aligned} \tag{51}$$

where internal stresses depend nonlinearly on strain rate through a viscosity

$$\begin{aligned}|\mathbf{D}(\boldsymbol{u})|^{n-1} &= (|\mathbf{D}_H(\boldsymbol{u}_H)|^2 + |\partial_z \boldsymbol{u}_H + \nabla_H u_z|^2/2 + |\partial_z u_z|^2)^{(n-1)/2} \\ &= (|\mathbf{D}_H(\boldsymbol{u}_H)|^2 + |\partial_z \boldsymbol{u}_H + O(\varepsilon)|^2/2 + |\,\mathrm{div}_H\, \boldsymbol{u}_H|^2)^{(n-1)/2}. \end{aligned} \tag{52}$$

The simple constant case $n = 1$ has been treated in the previous section. The cases $0 < n < 1$ and $n > 1$ are different due to different monotonocity properties of the stresses with respect to the deformation gradient $\boldsymbol{D}(\boldsymbol{u})$. The limit $n \to 0$ is singular: it yields a particular occurence of the Bingham model for viscoplastic fluids with a yield stress $|\boldsymbol{D}(\boldsymbol{u})| \neq 0 \Leftrightarrow |\boldsymbol{T}| > \frac{2}{\mathrm{Re}}$.

5.1. The fast flow regime

Let us look for a coherent approximation of the solutions to the initial BVP when (H1 – H2a – H3 – H4) hold, so that the simplifications of Section 3 also hold. Only the internal stresses are different in the present purely-viscous non-Newtonian case compared with the Newtonian case. And one can follow the same procedure until the construction of a correction. The question is: can we proceed, starting from the nonlinear version of (40), viz.

$$\frac{1}{\mathrm{Re}}(|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2+\frac{|\partial_z\boldsymbol{u}_H^1|^2}{2}+|\operatorname{div}_H\boldsymbol{u}_H^0|^2)^{(n-1)/2}\partial_z\boldsymbol{u}_H^1 = k\boldsymbol{u}_H^0\frac{b+h-z}{h}+O(\varepsilon^2)\,, \tag{53}$$

and define a correction $\boldsymbol{u}_H^1 = O(\varepsilon)$ using the same trick as in the Newtonian case, that is require $\int_b^{b+h}\boldsymbol{u}_H^1 = O(\varepsilon^3)$? To that aim, on requiring $\boldsymbol{u}_H^1 = O(\varepsilon^2)$ somewhere in the layer $z \in (b, b+h)$ (e.g. at bottom to not modify Navier friction condition at first order), it suffices to check that $\int_b^{b+h}\partial_z\boldsymbol{u}_H^1 = O(\varepsilon^2)$.

For $n \geq 1$ (shear-thickening fluids), define $\phi_a : x \to (x^2/2+a)^{(n-1)/2}x$ a function one-to-one and onto from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ so we rewrite (componentwise)

$$\partial_z\boldsymbol{u}_H^1 = \phi_a^{-1}(|\mathrm{Re}k\boldsymbol{u}_H^0(b+h-z)/h|)sg(\mathrm{Re}k\boldsymbol{u}_H^0(b+h-z)/h)$$

as a function of z parametrized by $\boldsymbol{u}_H^0$ through $a = |\mathbf{D}_H(\boldsymbol{u}_H^0)|^2+|\operatorname{div}_H\boldsymbol{u}_H^0|^2$. Notice that it holds $0 \leq \phi_a^{-1}(|\mathrm{Re}k\boldsymbol{u}_H^0(b+h-z)/h|) \leq |\mathrm{Re}k\boldsymbol{u}_H^0(b+h-z)/h|$ for $z \in (b, b+h)$, and on recalling that $\int_b^{b+h}|\mathrm{Re}k\boldsymbol{u}_H^0(b+h-z)/h|\,dz = O(\varepsilon^2)$ from the Netwonian case, it follows that a correction $\boldsymbol{u}_H^1$ can be constructed here. A coherent first-order approximation $\tilde{\boldsymbol{u}}_H^0$ is next defined from (42–43) where

(i) h^0 in viscous terms of (43) is replaced by[2] $\int_b^{b+h^0}\frac{|\mathrm{Re}k\boldsymbol{u}_H^0(b+h^0-z)/h^0|}{\phi_a^{-1}(|\mathrm{Re}k\boldsymbol{u}_H^0(b+h^0-z)/h^0|)}dz$

(ii) the friction term invokes the new value[3] of $\boldsymbol{u}_H|_{z=b}$ approximated at $O(\varepsilon^2)$.

This straightforwardly defines a coherent approximation insofar as the only equation which is different from the (coherent) Newtonian case is the equilibration of second-order viscous dissipation terms with friction at the bottom boundary, and a coherent correction has been constructed on purpose.

Note also that this reduced model obtained for $n \geq 1$ seems new to us. Though, it is not very practical (some terms are implicit) and may not be very useful for applications (shear-thickening fluids are not very common).

For $0 \leq n < 1$ (shear-thinning fluids), we cannot conclude with the strategy above. On invoking monotonicity again, let us then compare with $n \to 0$.

For $n = 0$, the point is to solve, for $\partial_z\boldsymbol{u}_H^1$, where $|\mathbf{D}_H(\boldsymbol{u}_H^0)| \neq 0$:

$$\frac{\partial_z\boldsymbol{u}_H^1}{\sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2+|\partial_z\boldsymbol{u}_H^1|^2/2+|\operatorname{div}_H\boldsymbol{u}_H^0|^2}} = \frac{\mathrm{Re}k\boldsymbol{u}_H^0}{h}(b+h-z)+O(\varepsilon)\,. \tag{54}$$

[2]This is a function of $\boldsymbol{u}_H^0$ and h^0 that one may obtain numerically after integration $\int_b^{b+h^0}\cdot\,dz$ by quadrature of the terms inside div_H.

[3]This is only known to be bounded above by $h^0\max_{z\in[b,b+h^0]}\partial_z\boldsymbol{u}_H^1 \leq \mathrm{Re}k\boldsymbol{u}_H^0h^0$.

But on the one hand, it should hold $|\mathbf{D}_H(\boldsymbol{u}_H^0)| = 0 \Leftrightarrow |\mathbf{D}_H(\boldsymbol{u}_H)| = 0$ whenever $\boldsymbol{u}_H^0$ does not vanish with ε, otherwise (54) contradicts coherence. And on the other hand, (54) has real-valued solutions $\partial_z \boldsymbol{u}_H^1$ that are compatible with the scaling implied by (H1 − H4) if, and only if, $0 < \boldsymbol{u}_H^0 < \sqrt{2}/(k\mathrm{Re})$ is satisfied (componentwise). Of course, the correction is then computed exactly, it reads

$$\boldsymbol{u}_H^1 = \sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}\,\frac{h}{\mathrm{Re}k|\boldsymbol{u}_H^0|}\left(2\sqrt{1-\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}h}(b+h-z)\right)^2}\right.$$
$$\left. + \sqrt{1-\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}}\right)^2} + \frac{\sqrt{2}}{\mathrm{Re}k|\boldsymbol{u}_H^0|}\arcsin\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}}\right)\right)\frac{\boldsymbol{u}_H^0}{|\boldsymbol{u}_H^0|}\,. \quad (55)$$

Let us recall that the case $n = 0$ models Bingham fluids, and is often formulated as a variational inequality (see e.g. [32]). Now, the reduced model for $n = 0$ can also be obtained as a variational inequality. Testing the initial problem with adequate functions $\boldsymbol{v}_H$ of $(x, y) \in \Omega$, one gets the following reduced problem similar to that in [23] after depth-averageing for $z \in (b, b + h)$

$$\partial_t h^0 + \operatorname{div}_H(h^0 \boldsymbol{u}_H^0) = 0\,, \quad (56)$$

$$\int_\Omega \left(\partial_t(h^0\boldsymbol{u}_H^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) + k(\boldsymbol{u}_H^0 + \boldsymbol{u}_H^1|_b)\right)\cdot(\boldsymbol{v}_H - \boldsymbol{u}_H^0) \quad (57)$$
$$+ \int_\Omega \frac{2}{\mathrm{Re}}\left|\mathbf{D}_H(\boldsymbol{v}_H) - \mathbf{D}_H(\boldsymbol{u}_H^0)\right| \geq$$
$$\int_\Omega \left(\frac{2}{\mathrm{Re}}\operatorname{div}_H\left(\beta\frac{\mathbf{D}_H(\boldsymbol{u}_H^0) + \operatorname{div}_H \boldsymbol{u}_H^0 \boldsymbol{I}}{\sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}}\right)\right)\cdot(\boldsymbol{v}_H - \boldsymbol{u}_H^0)$$
$$+ \int_\Omega \left(\gamma h^0 \nabla_H \Delta_H(b+h^0) + h^0 \boldsymbol{f}_H + f_z h^0 \nabla_H(b+h^0)\right)\cdot(\boldsymbol{v}_H - \boldsymbol{u}_H^0)$$

where, using the explicit profile (55), one can compute the friction term and the modified viscous coefficient $\beta = \int_b^{b+h^0}\left(1+\alpha(z)^2\right)^{-\frac{1}{2}} dz$ on invoking

$$\alpha^2 = \frac{h^0}{\mathrm{Re}k|\boldsymbol{u}_H^0|^2}\left|2\sqrt{1-\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}h^0}(b+h^0-z)\right)^2} + \sqrt{1-\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}}\right)^2} + \frac{\sqrt{2}}{\mathrm{Re}k|\boldsymbol{u}_H^0|}\arcsin\left(\frac{\mathrm{Re}k|\boldsymbol{u}_H^0|}{\sqrt{2}}\right)\right|^2.$$

But remember that the reduced model above (for the limit case $n \to 0$) breaks down when $\boldsymbol{u}_H^0 > \sqrt{2}/(k\mathrm{Re})$ while, at the same time, it has no meaning when $|\mathbf{D}_H(\boldsymbol{u}_H^0)| = 0 \Leftrightarrow |\boldsymbol{T}| < \frac{2}{\mathrm{Re}}$, which seems contradictory.

For $0 \leq n < 1$, one might finally be able to construct a correction that satisfies $\int_b^{b+h} \boldsymbol{u}_H^1 = O(\varepsilon^3)$ and define a coherent approximation of the full model, at least provided $\boldsymbol{u}_H^0 < \sqrt{2}/(k\mathrm{Re})$ holds where $\lim_{n\to 0}|\mathbf{D}_H(\boldsymbol{u}_H^0)| \neq 0$. One then obtains the same reduced model as that derived for $n \geq 1$, since one can still compute a correction in the same way; the limit $n \to 1$ of that model still coincides with the corrected viscous shallow water equations. But the flow regime required for consistency seems quite narrow.

Remark 3 (About viscoplastic non-Newtonian fluids).

The modelling of viscoplastic non-Newtonian fluids with a yield stress like in Bingham model above is still much debated, see e.g. [50]. In any case, Bingham law is a cornerstone of the viscoplastic modelling since it allows to mathematically investigate the concept of yield-stress and it is worth discussing. That is why we would also like to mention that the most usual form of Bingham law is not as above, but includes an additional viscous dissipative term, and is often thought as a particular case of the more general Herschel-Bulkley law

$$\boldsymbol{T} = \left(\frac{2}{Re}|\boldsymbol{D}(\boldsymbol{u})|^m + Bi\right)\frac{\boldsymbol{D}(\boldsymbol{u})}{|\boldsymbol{D}(\boldsymbol{u})|} \text{ if } \boldsymbol{D}(\boldsymbol{u}) \neq \boldsymbol{0}, \text{ then } |\boldsymbol{T}| \geq Bi, \text{ or } \boldsymbol{D}(\boldsymbol{u}) = \boldsymbol{0} \Leftrightarrow |\boldsymbol{T}| < Bi,$$

where this time we have denoted Bi a yield-stress independent of $\frac{2}{Re}$, the usual nondimensional constant for the ratio between the viscous dissipation and inertia. The standard Bingham law coincides with the case $m = 1$, while above we investigated the case $m = 0$ with $n = 0$.

For any m, the conclusion above needs to be modified as follows, provided one assumes $Bi \sim \varepsilon$ in order to perform our thin-layer reduction procedure. As above, one cannot go further than derive a reduced-model for the subdomains of the two-dimensional domain Ω where $|\mathbf{D}_H(\boldsymbol{u}_H^0)| \neq 0$ holds. And the problem still consists in computing a correction from a profile solution to

$$\left(\frac{2}{Re}|\boldsymbol{D}(\boldsymbol{u})|^m + Bi\right)\frac{\partial_z \boldsymbol{u}_H^1}{\sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\partial_z \boldsymbol{u}_H^1|^2/2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}} = \frac{k\boldsymbol{u}_H^0}{h}(b+h-z) + O(\varepsilon^2)\,, \tag{58}$$

an equation in $|\partial_z \boldsymbol{u}_H^1|$ unlikely to possess explicit solutions whatever $m > 0$.

Another viscoplastic paradigm has attracted much attention recently [37]: a Drucker-Prager yield criterion can replace the Von Mises one above, namely

$$\boldsymbol{T} = \left(\frac{2}{Re}|\boldsymbol{D}(\boldsymbol{u})|^m + pBi\right)\frac{\boldsymbol{D}(\boldsymbol{u})}{|\boldsymbol{D}(\boldsymbol{u})|} \text{ if } \boldsymbol{D}(\boldsymbol{u}) \neq \boldsymbol{0}, \text{ then } |\boldsymbol{T}| \geq pBi, \text{ or } \boldsymbol{D}(\boldsymbol{u}) = \boldsymbol{0} \Leftrightarrow |\boldsymbol{T}| < pBi. \tag{59}$$

Note that it is not necessary to assume $Bi = O(\varepsilon)$ then since $p = O(\varepsilon)$. In particular, when $Re \to \infty$, the correction to the velocity profile should satisfy

$$pBi\frac{\partial_z \boldsymbol{u}_H^1}{\sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\partial_z \boldsymbol{u}_H^1|^2/2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}} = \frac{k\boldsymbol{u}_H^0}{h}(b+h-z) + O(\varepsilon^2)\,, \tag{60}$$

where $p = f_z(z-(b+h)) - \gamma\Delta_H(b+h) + T_{zz} + O(\varepsilon^2)$ from (15)*. On noting* (59)*,*

$$p\left(1 + Bi\frac{\operatorname{div}_H(\boldsymbol{u}_H^0)}{\sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\partial_z \boldsymbol{u}_H^1|^2/2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}}\right) = f_z(z-(b+h)) - \gamma\Delta_H(b+h) + O(\varepsilon^2)$$

plugged into (60) *yields an algebraic equation for $\partial_z \boldsymbol{u}_H^1$ at any $z \in (b, b+h)$*

$$\frac{Bi\partial_z \boldsymbol{u}_H^1\left(f_z(z-(b+h)) - \gamma\Delta_H(b+h)\right)}{Bi\operatorname{div}_H(\boldsymbol{u}_H^0) + \sqrt{|\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + |\partial_z \boldsymbol{u}_H^1|^2/2 + |\operatorname{div}_H \boldsymbol{u}_H^0|^2}} = \frac{k\boldsymbol{u}_H^0}{h}(b+h-z) + O(\varepsilon^2)\,. \tag{61}$$

In the case $\gamma = 0$ (no surface tension), the formula becomes much easier

$$\left(\frac{1}{2} - \left(\frac{hBif_z}{k\boldsymbol{u}_H^0}\right)^2\right)|\partial_z \boldsymbol{u}_H^1|^2 - 2Bi\,\mathrm{div}_H(\boldsymbol{u}_H^0)\left(\frac{hBif_z}{k\boldsymbol{u}_H^0}\right)|\partial_z \boldsymbol{u}_H^1| + |\mathbf{D}_H(\boldsymbol{u}_H^0)|^2 + (1 - Bi^2)|\,\mathrm{div}_H\,\boldsymbol{u}_H^0|^2 + O(\varepsilon^2) = 0 \quad (62)$$

and one can compute explicitly the correction. So the solution to (62) *allows one to define an admissible velocity correction, and thus also a coherent approximation of the full model through the reduced model, as soon as the sole requirement $|\mathbf{D}_H(\boldsymbol{u}_H^0)| \neq 0 \Leftrightarrow |\mathbf{D}_H(\boldsymbol{u}_H)| \neq 0$ is satisfied here (a condition that unfortunately remains difficult to predict or analyze here ; in particular, we are not aware of a simpler reformulation of this model as a variational inequality).*

5.2. The slow flow regime

Assuming (H1 – H2b – H3 – H4) we follow the same procedure as in the Newtonian case. First we obtain a nonlinear version of (46)

$$\frac{1}{\mathrm{Re}}(|\partial_z \boldsymbol{u}_H|^2/2)^{(n-1)/2}\partial_z \boldsymbol{u}_H = \boldsymbol{f}_H\,(z - (b+h)) + O(\varepsilon^2) \quad (63)$$

and with the friction boundary condition at $z = b$, this next yields

$$\boldsymbol{u}_H = \left(\mathrm{Re}2^{\frac{n-1}{2}}\boldsymbol{f}_H\right)^{\frac{1}{n}}\left(\frac{(z-(b+h))^{\frac{n+1}{n}} - h^{\frac{n+1}{n}}}{\frac{n+1}{n}}\right) - \boldsymbol{f}_H\frac{h}{k} + O(\varepsilon^{1+\frac{2}{n}})\,. \quad (64)$$

We can finally get an autonomous equation for $h^0 = h + O(\varepsilon^2)$ from the continuity equation $\partial_t h + \mathrm{div}_H \int_b^{b+h} \boldsymbol{u}_H = 0$ and the approximation

$$\int_b^{b+h} \boldsymbol{u}_H = \left(\mathrm{Re}2^{\frac{n-1}{2}}\boldsymbol{f}_H\right)^{\frac{1}{n}}\left(\frac{2n+1}{n+1}h^{\frac{2n+1}{n}}\right) - \boldsymbol{f}_H\frac{h^2}{k} + O(\varepsilon^{2+\frac{2}{n}})\,. \quad (65)$$

This coincides with the viscous limit recently derived in [33], though with another mathetically-inclined viewpoint and a slightly different scaling (the term h^2/k is absent in particular, as a result of the no-slip limit $k \to \infty$). It holds for all power-law fluids $n > 0$, but note that the quality of approximation increases with n in the shear-thinning case while it decreases in the shear-thickening case.

6. Application to viscoelastic non-Newtonian fluids

There are numerous models for viscoelastic non-Newtonian fluids, with various definitions of the extra-stress $\boldsymbol{\tau}$ in $\boldsymbol{T} = 2\eta_s \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{\tau}$. We concentrate here on one prototypical model among *differential* constitutive equations for $\boldsymbol{\tau}$, the Upper-Convected Maxwell (UCM) equations [15],

$$D_t\boldsymbol{\tau} = (\nabla\boldsymbol{u})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla\boldsymbol{u})^T + \frac{1}{\lambda}(2\eta_p \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\tau}) \qquad \text{in } \mathcal{D}(t), \quad (66)$$

where λ is interpreted as a characteristic relaxation time for elastic dilute molecules and η_p as a viscosity. There are many extensions to the UCM equations, which one also often writes using the total (kinematic) viscosity $\eta = \eta_s + \eta_p$ and the retardation time $\lambda(1-\theta) \le \lambda$ where $\theta = \eta_p/\eta \in (0,1)$

$$\begin{cases} D_t \boldsymbol{u} = -\nabla p + \operatorname{div}(2\eta(1-\theta)\boldsymbol{D}(\boldsymbol{u})) + \operatorname{div}\boldsymbol{\tau} + \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} = 0 \\ \lambda(D_t\boldsymbol{\tau} - (\nabla\boldsymbol{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla\boldsymbol{u})^T) = 2\eta\theta\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\tau} \end{cases} \quad \text{in } \mathcal{D}(t)\,. \tag{67}$$

A simple one for instance combines the power-law and the UCM models $\boldsymbol{T} = 2\eta_s|\boldsymbol{D}(\boldsymbol{u})|^{n-1}\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{\tau}$, see [46]. One can also use nonlinear versions of the relaxation term in the right-hand side of (66), see [49]. Interestingly, a model that also builds on such nonlinear possible extensions of the UCM model [53] has been proposed recently for *elastoviscoplastic* materials (which, we recall, are still much debated). In any case (66) contains the kinematic essence of most *differential* constitutive equations (material frame indifference for tensors) and we postpone the discussion of other models to Remark 4 (and future works).

We nondimensionalize (67) introducing the Deborah number $\mathrm{De} = \lambda/T$, and

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{HH} & \boldsymbol{T}_{Hz} \\ \boldsymbol{T}_{Hz}^T & T_{zz} \end{pmatrix} = \frac{1-\theta}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z\boldsymbol{u}_H + \nabla_H u_z \\ (\partial_z\boldsymbol{u}_H + \nabla_H u_z)^T & 2\partial_z u_z \end{pmatrix} + \begin{pmatrix} \boldsymbol{\tau}_{HH} & \boldsymbol{\tau}_{Hz} \\ \boldsymbol{\tau}_{Hz}^T & \tau_{zz} \end{pmatrix}$$

so the extra-stress $\boldsymbol{\tau}$ satisfies the nondimensional UCM equations

$$\mathrm{De}\left(D_t\boldsymbol{\tau} - (\nabla\boldsymbol{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\nabla\boldsymbol{u})^T\right) = \frac{2\theta}{\mathrm{Re}}\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{\tau}\,. \tag{68}$$

Note that the cases $\mathrm{De} = O(\varepsilon)$ are a priori not the most physically interesting because they lead us back to a purely-viscous Newtonian extra-stress when $\varepsilon \to 0$, while, on the contrary, $\mathrm{De} \to \infty$ is a well-known, difficult and interesting limit toward elastic solid behaviours. In the following, we use the reformulation of (68) with the also well-known *conformation tensor* variable $\boldsymbol{\sigma} = \boldsymbol{I} + \frac{\mathrm{DeRe}}{\theta}\boldsymbol{\tau}$, which is solution to an evolution equation using the single scalar parameter De

$$\mathrm{De}\left(D_t\boldsymbol{\sigma} - (\nabla\boldsymbol{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla\boldsymbol{u})^T\right) = \boldsymbol{I} - \boldsymbol{\sigma}\,. \tag{69}$$

The Weissenberg number $\mathrm{Wi} = \mathrm{DeRe}/\theta$ appears in Navier-Stokes (10) through

$$\boldsymbol{T} = \frac{1-\theta}{\mathrm{Re}} \begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & \partial_z\boldsymbol{u}_H + O(\varepsilon) \\ (\partial_z\boldsymbol{u}_H + O(\varepsilon))^T & -2\operatorname{div}_H\boldsymbol{u}_H \end{pmatrix} + \frac{\theta}{\mathrm{ReDe}} \begin{pmatrix} \boldsymbol{\sigma}_{HH} - \boldsymbol{I}_H & \boldsymbol{\sigma}_{Hz} \\ \boldsymbol{\sigma}_{Hz}^T & \sigma_{zz} - 1 \end{pmatrix}, \tag{70}$$

where, recalling Section 3, we used the continuity equation, $h \sim \varepsilon$ and (H1) in

$$\nabla\boldsymbol{u} = \begin{pmatrix} \nabla_H\boldsymbol{u}_H & \partial_z\boldsymbol{u}_H \\ (\nabla_H u_z)^T & \partial_z u_z \end{pmatrix} = \begin{pmatrix} \nabla_H\boldsymbol{u}_H & \partial_z\boldsymbol{u}_H \\ O(\varepsilon) & -\operatorname{div}_H\boldsymbol{u}_H \end{pmatrix}.$$

We recall that for physical reasons[4], the conformation tensor should always be *positive-definite*, and indeed remains so as long as it is initially and solutions to (69) are smooth enough (see e.g. [22]).

[4]For the model to be consistent with the usual thermodynamics principles: see e.g. [57].

From now on, recalling Section 3, it is natural to assume $\boldsymbol{\sigma}_{HH}$ and σ_{zz} are not only bounded but also have same scaling. Then, on noting that (69) reads

$$\mathrm{De}\left(D_t\boldsymbol{\sigma}_{HH} - (\nabla_H \boldsymbol{u}_H)\boldsymbol{\sigma}_{HH} - \boldsymbol{\sigma}_{HH}(\nabla_H \boldsymbol{u}_H)^T - \boldsymbol{\sigma}_{Hz} \otimes \partial_z \boldsymbol{u}_H - \partial_z \boldsymbol{u}_H \otimes \boldsymbol{\sigma}_{Hz}\right) = \boldsymbol{\sigma}_{HH} - \boldsymbol{I} \tag{71a}$$

$$\mathrm{De}\left(D_t\boldsymbol{\sigma}_{Hz} - (\nabla_H \boldsymbol{u}_H)\boldsymbol{\sigma}_{Hz} - \boldsymbol{\sigma}_{HH}(\nabla_H u_z) - \boldsymbol{\sigma}_{Hz}\partial_z u_z - \partial_z \boldsymbol{u}_H \sigma_{zz}\right) = \boldsymbol{\sigma}_{Hz} \tag{71b}$$

$$\mathrm{De}\left(D_t\sigma_{zz} - 2\boldsymbol{\sigma}_{Hz} \cdot \nabla_H u_z - 2\sigma_{zz}\partial_z u_z\right) = \sigma_{zz} - 1\,, \tag{71c}$$

it stems from (71a) and (71b) that (H4) : $\partial_z \boldsymbol{u}_H = O(1)$ is also as natural (for boundedness) in viscoelastic non-Newtonian fluids as in purely viscous (Newtonian and non-Newtonian) fluids. Under (H4), one then obtains with $\mathrm{De} \sim 1$

$$\mathrm{De}\left(D_t\boldsymbol{\sigma}_{HH} - (\nabla_H \boldsymbol{u}_H^0)\boldsymbol{\sigma}_{HH} - \boldsymbol{\sigma}_{HH}(\nabla_H \boldsymbol{u}_H^0)^T - \boldsymbol{\sigma}_{Hz} \otimes \partial_z \boldsymbol{u}_H^1 - \partial_z \boldsymbol{u}_H^1 \otimes \boldsymbol{\sigma}_{Hz}\right) = \boldsymbol{\sigma}_{HH} - \boldsymbol{I} + O(\varepsilon) \tag{72a}$$

$$\mathrm{De}\left(D_t\boldsymbol{\sigma}_{Hz} - (\nabla_H \boldsymbol{u}_H^0)\boldsymbol{\sigma}_{Hz} + \boldsymbol{\sigma}_{Hz}\,\mathrm{div}_H\, \boldsymbol{u}_H^0 - \partial_z \boldsymbol{u}_H^1 \sigma_{zz}\right) = \boldsymbol{\sigma}_{Hz} + O(\varepsilon) \tag{72b}$$

$$\mathrm{De}\left(D_t\sigma_{zz} + 2\sigma_{zz}\,\mathrm{div}_H\, \boldsymbol{u}_H^0\right) = \sigma_{zz} - 1 + O(\varepsilon) \tag{72c}$$

from (71a–71b–71c), for any first-order approximation $\boldsymbol{u}_H^0 = \boldsymbol{u}_H + O(\varepsilon)$ with a flat profile, possibly corrected by some $\boldsymbol{u}_H^1 = O(\varepsilon)$.

6.1. The fast flow regime

Like in the previous cases, we obtain an inertial limit when one specifies (H2) as (H2a) : $k \sim \varepsilon$. On the contrary, to coherently use $\boldsymbol{T}_{Hz} = O(\varepsilon)$ for BCs (17) and (18) in Section 3 with

$$\boldsymbol{T}_{Hz} = \frac{1}{\mathrm{Re}}\left((1-\theta)\partial_z \boldsymbol{u}_H^1 + \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}\right)\,, \tag{73}$$

we should now further assume, in addition to (H4),

(i) either (H3) : $\mathrm{Re} \sim \varepsilon^{-1}$ like in the Newtonian case,

(ii) or (H5a) : $1-\theta \sim \varepsilon$, plus either (H6a) : $\boldsymbol{\sigma}_{Hz} = O(\varepsilon)$ or (H6c) : $\mathrm{De} \sim \varepsilon^{-1}$,

(iii) or (H5b) : $\partial_z \boldsymbol{u}_H = O(\varepsilon)$ (which is stronger than (H4)) plus either (H6a), or (H6b) : $\theta \sim \varepsilon$, or (H6c).

Note that in absence of other assumptions, $\mathrm{De} \sim 1$ and $\theta \sim 1$ shall be simply taken as constants.

6.1.1. Small internal stresses

Under assumptions (H1 − H2a − H4 − H3), like in the Newtonian case, first-order approximations $(h^0, \boldsymbol{u}_H^0)$ solution to (34–35) are not necessarily coherent with BCs (17) and (18) and the point is how to approximately compute $\boldsymbol{T}_{Hz}$. Introducing a correction $\boldsymbol{u}_H^1$ satisfying

$$\frac{1}{\mathrm{Re}}\left((1-\theta)\partial_z \boldsymbol{u}_H^1 + \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}\right) = k\boldsymbol{u}_H^0\frac{b+h-z}{h} + O(\varepsilon^2)\,, \tag{74}$$

one would then like to coherently replace (34–35) by a reduced model invoking the depth-averaged horizontal momentum equation just like in the Newtonian case, plus simplified UCM equations to close the system. Now, let us rewrite (74)

$$\partial_z \boldsymbol{u}_H^1 = \frac{1}{1-\theta}\left(\mathrm{Re}k\boldsymbol{u}_H^0 \frac{b+h-z}{h} - \theta \frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}\right) + O(\varepsilon)\,. \tag{75}$$

Then a coherent reduced model is obtained as usual after closing the second-order truncation of the horizontal momentum equation, using the same trick $\int_b^{b+h} \boldsymbol{u}_H^1 = O(\varepsilon^3)$ as in the Newtonian case. Assuming, for the sake of simplicity,

$$(\mathrm{H7a}) : \partial_z \boldsymbol{\sigma}_{HH}, \partial_z \sigma_{zz} = O(1) \qquad (\mathrm{H7b}) : \partial_z \boldsymbol{\sigma}_{Hz} = O(1)$$

a profile can be computed explicitly from (75) such that $\int_b^{b+h} \boldsymbol{u}_H^1 = O(\varepsilon^3)$ holds

$$\boldsymbol{u}_H^1 = \frac{1}{1-\theta}\left(\frac{\mathrm{Re}k}{2h}\boldsymbol{u}_H^0\left(\frac{h^2}{3} - (b+h-z)^2\right) - \frac{\theta}{2}\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\left(z-(h+2b)\right)\right)\,, \tag{76}$$

and the reduced model coherent with (H1 – H2a – H4 – H3 – H7) finally reads

$$\partial_t h^0 + \mathrm{div}_H(h^0 \boldsymbol{u}_H^0) = 0 \tag{77a}$$

$$\partial_t(h^0\boldsymbol{u}_H^0) + \mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{u}_H^0) + k\boldsymbol{u}_H^0\left(1 - \frac{\mathrm{Re}}{(1-\theta)}\frac{kh^0}{3}\right) + k\boldsymbol{\sigma}_{Hz}^0\frac{\mathrm{Re}}{(1-\theta)}\frac{\theta(b+h^0)}{2\mathrm{De}}$$
$$= \frac{2(1-\theta)}{\mathrm{Re}}\,\mathrm{div}_H\left(h^0\left(\mathbf{D}_H(\boldsymbol{u}_H^0) + \mathrm{div}_H\,\boldsymbol{u}_H^0\boldsymbol{I}\right)\right) + \frac{\theta}{\mathrm{ReDe}}\,\mathrm{div}_H\left(h^0(\boldsymbol{\sigma}_{HH}^0 - \sigma_{zz}^0\boldsymbol{I})\right)$$
$$+ \left(h^0\boldsymbol{f}_H + f_z h^0\nabla_H(b+h^0)\right) + \gamma h^0 \nabla_H\Delta_H(b+h^0) \tag{77b}$$

$$\partial_t(h^0\boldsymbol{\sigma}_{HH}^0) + \mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{\sigma}_{HH}^0) = h^0\left((\nabla_H\boldsymbol{u}_H^0)\boldsymbol{\sigma}_{HH}^0 + \boldsymbol{\sigma}_{HH}^0(\nabla_H\boldsymbol{u}_H^0)^T\right)$$
$$+ h^0\frac{1}{1-\theta}\left(\boldsymbol{\sigma}_{Hz}^0\otimes\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0 - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right) + \left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0 - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)\otimes\boldsymbol{\sigma}_{Hz}^0\right)$$
$$+ h^0\frac{1}{\mathrm{De}}(\boldsymbol{\sigma}_{HH}^0 - \boldsymbol{I}) \tag{77c}$$

$$\partial_t(h^0\boldsymbol{\sigma}_{Hz}^0) + \mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{\sigma}_{Hz}^0) = h^0\left(\nabla_H\boldsymbol{u}_H^0\boldsymbol{\sigma}_{Hz}^0 - \mathrm{div}_H\,\boldsymbol{u}_H^0\boldsymbol{\sigma}_{Hz}^0\right)$$
$$+ h^0\frac{1}{1-\theta}\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0 - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)\sigma_{zz}^0 + h^0\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0 \tag{77d}$$

$$\partial_t(h^0\sigma_{zz}^0) + \mathrm{div}_H(h^0\boldsymbol{u}_H^0\sigma_{zz}^0) = 2h^0\sigma_{zz}^0\,\mathrm{div}_H\,\boldsymbol{u}_H^0 + h^0\frac{1}{\mathrm{De}}(\sigma_{zz}^0 - 1)\,. \tag{77e}$$

This seems to be a new model. In particular, it was not identified in our previous work [18] that focused on the case $\theta = 1$. (In [18], one could not easily derive an expression for $\partial_z \boldsymbol{u}_H$: without linking shear strain with stress like (74), a coherent approximation of (72b) like (77d) is not obviously computed.)

Note that one retrieves the standard viscous shallow water (our reduced model for the standard Navier-Stokes equations) in the limit $\theta \to 0$ (prior or subsequent to $\varepsilon \to 0$; i.e. the two formal limits commute here), plus UCM equations that then become simply enslaved transport equations for a material tensor (without feedback in the momentum equation). The limit $\theta \to 1$ is unclear here, but we next assume $\theta = 1 + O(\varepsilon)$ (then with $\mathrm{Re} \sim 1$), and will be able to derive the limit when $\boldsymbol{\sigma}_{Hz}/\mathrm{De} = O(\varepsilon)$.

6.1.2. Small viscous internal stresses: High-Weissenberg limit

Under assumptions (H1 − H2a − H5a − H6a), a non-vanishing approximation of $\boldsymbol{\sigma}_{Hz}$ can be coherently constructed from (71b) only if (H5b) holds. Indeed, $\sigma_{zz} = O(\varepsilon)$ is not possible by (71c) and this is not coherent with (74) and the horizontal momentum equation unless $\boldsymbol{u}_H^0 \to 0$. So let us only consider (H1 − H2a − H5a − H6c), plus (H7) for the sake of simplicity. This leads to

$$\partial_t h^0 + \mathrm{div}_H(h^0 \boldsymbol{u}_H^0) = 0 \tag{78a}$$

$$\begin{aligned}\partial_t(h^0\boldsymbol{u}_H^0)+\mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{u}_H^0)+k\boldsymbol{u}_H^0\left(1-\frac{\mathrm{Re}}{(1-\theta)}\frac{kh^0}{3}\right)+k\boldsymbol{\sigma}_{Hz}^0\frac{\mathrm{Re}}{(1-\theta)}\frac{\theta(b+h^0)}{2\mathrm{De}}\\ = \frac{2(1-\theta)}{\mathrm{Re}}\,\mathrm{div}_H\left(h^0\left(\mathbf{D}_H(\boldsymbol{u}_H^0)+\mathrm{div}_H\,\boldsymbol{u}_H^0\boldsymbol{I}\right)\right)+\frac{\theta}{\mathrm{ReDe}}\,\mathrm{div}_H\left(h^0(\boldsymbol{\sigma}_{HH}^0-\sigma_{zz}^0\boldsymbol{I})\right)\\ +\left(h^0\boldsymbol{f}_H+f_zh^0\nabla_H(b+h^0)\right)+\gamma h^0\nabla_H\Delta_H(b+h^0)\end{aligned} \tag{78b}$$

$$\begin{aligned}\partial_t(h^0\boldsymbol{\sigma}_{HH}^0)+\mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{\sigma}_{HH}^0)=h^0\left((\nabla_H\boldsymbol{u}_H^0)\boldsymbol{\sigma}_{HH}^0+\boldsymbol{\sigma}_{HH}^0(\nabla_H\boldsymbol{u}_H^0)^T\right)\\ +h^0\frac{1}{1-\theta}\left(\boldsymbol{\sigma}_{Hz}^0\otimes\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0-\theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)+\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0-\theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)\otimes\boldsymbol{\sigma}_{Hz}^0\right)\end{aligned} \tag{78c}$$

$$\begin{aligned}\partial_t(h^0\boldsymbol{\sigma}_{Hz}^0)+\mathrm{div}_H(h^0\boldsymbol{u}_H^0\otimes\boldsymbol{\sigma}_{Hz}^0)=h^0(\nabla_H\boldsymbol{u}_H^0)\boldsymbol{\sigma}_{Hz}^0-h^0\boldsymbol{\sigma}_{Hz}^0\,\mathrm{div}_H\,\boldsymbol{u}_H^0\\ +h^0\frac{1}{1-\theta}\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0-\theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)\sigma_{zz}^0\end{aligned} \tag{78d}$$

$$\partial_t(h^0\sigma_{zz}^0)+\mathrm{div}_H(h^0\boldsymbol{u}_H^0\sigma_{zz}^0)=2h^0\sigma_{zz}^0\,\mathrm{div}_H\,\boldsymbol{u}_H^0 \tag{78e}$$

whose solutions are coherent with the initial BVP when (H1 − H2a − H5a − H6c − H7ab) holds.

The model (6.13) formally coincides with the limit $1/\mathrm{De} \to 0$, $\theta \to 1$ of the previous section (provided $k/(1-\theta)$ and $1/\mathrm{De}(1-\theta)$ remain bounded), where the elastic relaxation time is infinitely long as well as the retardation time associated with the purely viscous term. It is some kind of "High-Weissenberg limit", where the UCM model suffers from deficiencies (see e.g. [22], and Remark 4 for repair suggestions). Observe that our reduced model has essentially lost the terms corresponding to the physical relaxation to thermodynamical equilibrium !

6.1.3. Small viscous internal shear stresses

Under assumptions (H1−H2a−H5b), the motion-by-slice is stronger than the usual one. This a priori restricts the regimes of validity for the reduced model (even if solutions exist beyond the validity regime, they would not necessarily define coherent approximations of the initial BVP). It implies

$$\boldsymbol{u}_H(t,x,y,z) = \boldsymbol{u}_H^0(t,x,y) + O(\varepsilon^2) \tag{79}$$

so that the correction $\boldsymbol{u}_H^1$ to $\boldsymbol{u}_H^0$ is of higher-order than usual ones and does not show up in the horizontal momentum equation if, on the other hand, the extra-stress terms can be computed coherently.

Now, under (H1 − H2a − H5b − H6a − H7ab) – (H7) for the sake of simplicity – one obtains the following reduced model, coherent with the initial BVP

$$\partial_t h^0 + \operatorname{div}_H(h^0 \boldsymbol{u}_H^0) = 0 \tag{80a}$$

$$\begin{aligned}
&\partial_t(h^0\boldsymbol{u}_H^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{u}_H^0) + k\boldsymbol{u}_H^0 \\
&= \frac{2(1-\theta)}{\mathrm{Re}} \operatorname{div}_H \left(h^0 \left(\mathbf{D}_H(\boldsymbol{u}_H^0) + \operatorname{div}_H \boldsymbol{u}_H^0 \boldsymbol{I}\right)\right) + \frac{\theta}{\mathrm{ReDe}} \operatorname{div}_H \left(h^0(\boldsymbol{\sigma}_{HH}^0 - \sigma_{zz}^0 \boldsymbol{I})\right) \\
&\qquad + \left(h^0 \boldsymbol{f}_H + f_z h^0 \nabla_H (b + h^0)\right) + \gamma h^0 \nabla_H \Delta_H (b + h^0)
\end{aligned} \tag{80b}$$

$$\begin{aligned}
\partial_t(h^0\boldsymbol{\sigma}_{HH}^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{\sigma}_{HH}^0) &= h^0 \left((\nabla_H \boldsymbol{u}_H^0)\boldsymbol{\sigma}_{HH}^0 + \boldsymbol{\sigma}_{HH}^0 (\nabla_H \boldsymbol{u}_H^0)^T\right) \\
&\quad + h^0 \frac{1}{\mathrm{De}}(\boldsymbol{\sigma}_{HH}^0 - \boldsymbol{I})
\end{aligned} \tag{80c}$$

$$\begin{aligned}
&\partial_t(h^0\boldsymbol{\sigma}_{Hz}^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0 \otimes \boldsymbol{\sigma}_{Hz}^0) = h^0 \left((\nabla_H \boldsymbol{u}_H^0)\boldsymbol{\sigma}_{Hz}^0 - h^0 \boldsymbol{\sigma}_{Hz}^0 \operatorname{div}_H \boldsymbol{u}_H^0\right) \\
&\quad + h^0 \boldsymbol{\sigma}_{HH}^0 \left(\nabla_H(\boldsymbol{u}_H^0 \cdot \nabla_H b) + (\operatorname{div}_H \boldsymbol{u}_H^0)\nabla_H b - \frac{1}{2} h^0 \nabla_H \operatorname{div}_H \boldsymbol{u}_H^0\right) \\
&\qquad + h^0 \frac{1}{1-\theta}\left(\frac{\mathrm{Re}k}{2}\boldsymbol{u}_H^0 - \theta \frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0\right)\sigma_{zz}^0 + h^0 \frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}^0
\end{aligned} \tag{80d}$$

$$\partial_t(h^0\sigma_{zz}^0) + \operatorname{div}_H(h^0\boldsymbol{u}_H^0\sigma_{zz}^0) = 2h^0\sigma_{zz}^0 \operatorname{div}_H \boldsymbol{u}_H^0 + h^0 \frac{1}{\mathrm{De}}(\sigma_{zz}^0 - 1)\,. \tag{80e}$$

where, contrary to (77a–77b–77c–77d–77e) or its “High-Weissenberg limit” (78a–78b–78c–78d–78e), the shear component $\boldsymbol{\sigma}_{Hz}$ of the viscoelastic stress decouples from the autonomous system of equations (80a–80b–80c–80e) and is simply computed as a post-processed solution to (80d) enslaved through $\boldsymbol{u}_H^0$. (In (80d), we have used (74) for the vertical derivative of the horizontal velocity, and the approximate vertical velocity $u_z^0 = u_z + O(\varepsilon^2)$ reconstructed from $\boldsymbol{u}_H^0$, the continuity equation and the impermeability condition at the bottom excatly like in the Newtonian case, so (80d) is coherent with a first-order approximation $\boldsymbol{\sigma}_{Hz}^0 = \boldsymbol{\sigma}_{Hz} + O(\varepsilon^2)$.)

The latter reduced model is a two-dimensional extension of the one-dimensional model derived in [18] when $\theta = 1$, $k = 0$ specifically. We recover the case of pure-slip condition $k = 0$ at bottom boundary for $\theta \in [0, 1)$ straightforwardly by taking the limit $k \to 0$ in the system above. The inviscid limit case $\theta \to 1$ can be treated assuming (H5a) : $1 - \theta \sim \varepsilon$ along with $k = O(\varepsilon^2)$, like for Newtonian fluids, see also the discussion in [20].

When one assumes (H6b − H7ab) in addition to (H1 − H2a − H5b), one straightforwardly obtains the same autonomous system of equations as in the reduced model with (H6a), that is (80a–80b–80c–80e). A coherent first-order approximation without even assuming any scaling for $\boldsymbol{\sigma}$ is then define ! (The coefficient θ is then only responsible for the small scale of the whole tensor.) Moreover, the shear approximation $\boldsymbol{\sigma}^0_{Hz} = \boldsymbol{\sigma}_{Hz} + O(\varepsilon)$ need not be smaller than elongational components of the stress tensor. It is found as a solution to

$$\mathrm{De}\left(D_t\boldsymbol{\sigma}^0_{Hz} - (\nabla_H \boldsymbol{u}^0_H)\boldsymbol{\sigma}^0_{Hz} + \boldsymbol{\sigma}^0_{Hz}\operatorname{div}_H \boldsymbol{u}^0_H\right) = \boldsymbol{\sigma}^0_{Hz}\,. \tag{81}$$

Last, (H1 − H2a − H5b − H6c − H7ab) yields a reduced model coherent with a "High-Weissenberg-limit" approximation of the initial BVP, that is

$$\partial_t h^0 + \operatorname{div}_H(h^0 \boldsymbol{u}^0_H) = 0 \tag{82a}$$

$$\begin{aligned}&\partial_t(h^0\boldsymbol{u}^0_H) + \operatorname{div}_H(h^0\boldsymbol{u}^0_H \otimes \boldsymbol{u}^0_H) + k\boldsymbol{u}^0_H \\ &= \frac{2(1-\theta)}{\mathrm{Re}}\operatorname{div}_H\left(h^0\left(\mathbf{D}_H(\boldsymbol{u}^0_H) + \operatorname{div}_H \boldsymbol{u}^0_H \boldsymbol{I}\right)\right) + \frac{\theta}{\mathrm{ReDe}}\operatorname{div}_H\left(h^0(\boldsymbol{\sigma}^0_{HH} - \sigma^0_{zz}\boldsymbol{I})\right) \\ &\quad + \left(h^0\boldsymbol{f}_H + f_z h^0 \nabla_H(b+h^0)\right) + \gamma h^0 \nabla_H \Delta_H(b+h^0)\end{aligned} \tag{82b}$$

$$\partial_t(h^0\boldsymbol{\sigma}^0_{HH}) + \operatorname{div}_H(h^0\boldsymbol{u}^0_H \otimes \boldsymbol{\sigma}^0_{HH}) = h^0(\nabla_H\boldsymbol{u}^0_H)\boldsymbol{\sigma}^0_{HH} + \boldsymbol{\sigma}^0_{HH}(\nabla_H\boldsymbol{u}^0_H)^T \tag{82c}$$

$$\partial_t(h^0\boldsymbol{\sigma}^0_{Hz}) + \operatorname{div}_H(h^0\boldsymbol{u}^0_H \otimes \boldsymbol{\sigma}^0_{Hz}) = (\nabla_H\boldsymbol{u}^0_H)\boldsymbol{\sigma}^0_{Hz} - \boldsymbol{\sigma}^0_{Hz}\operatorname{div}_H \boldsymbol{u}^0_H \tag{82d}$$

$$\partial_t(h^0\sigma^0_{zz}) + \operatorname{div}_H(h^0\boldsymbol{u}^0_H\sigma^0_{zz}) = 2h^0\sigma^0_{zz}\operatorname{div}_H \boldsymbol{u}^0_H\,. \tag{82e}$$

It is remarkable that no correction to the flat profile is necessary under assumption (H5b) (even if a profile can be reconstructed afterwards from (76)), whereas the presence of purely (Newtonian) viscous forces is in turn hardly seen but in dissipation terms when one enforces (H5b) instead of (H5a). Furthermore, requiring the velocity to have a flat profile (79) is thus a priori a very strong limit for the applicability of our reduced models to real flows. This may however be particularly interesting for the cases where the normal stress differences are large, since the stress (70) then reads

$$\boldsymbol{T} = \frac{1-\theta}{\mathrm{Re}}\begin{pmatrix} 2\mathbf{D}_H(\boldsymbol{u}_H) & O(\varepsilon) \\ O(\varepsilon) & -2\operatorname{div}_H \boldsymbol{u}_H \end{pmatrix} + \frac{\theta}{\mathrm{ReDe}}\begin{pmatrix} \boldsymbol{\sigma}_{HH} - \boldsymbol{I}_H & \boldsymbol{\sigma}_{Hz} \\ \boldsymbol{\sigma}^T_{Hz} & \sigma_{zz} - 1 \end{pmatrix}, \tag{83}$$

where either (H6b) : $\theta \sim \varepsilon$, eor (H8) : $\boldsymbol{\sigma}_{HH} = \boldsymbol{I} + O(\varepsilon)$, $\sigma_{zz} = 1 + O(\varepsilon)$, $\boldsymbol{\sigma}_{Hz} = O(\varepsilon^2)$ (a stronger assumption necessary when starting with (H6a) : $\boldsymbol{\sigma}_{Hz} \sim \varepsilon$) or (H6c) : $\mathrm{De} \sim \varepsilon^{-1}$ holds but the viscous stretch need not be scaled even though viscoelastic components are always small.

To conclude this section, note that even though some reduced models have been identified in the High-Weissenberg limit regime (H6c) : De $\sim \varepsilon^{-1}$ where the model is questionable, we have obtained otherwise two main reduced models – the closed systems of equations (77a–77b–77c–77d–77e) and (82a–82b–82c–82e) – whose solutions define coherent approximations of the initial BVP in physically sensible regimes. It would be interesting to numerically simulate the first one, which has not been done yet to our knowledge, and compare it to the two-dimenionsal extension of the model in [18]. Note in particular that shear effects are then not necessarily small in comparison with elongational/compression effects, which we suspected to be a problem for the applicability of the second reduced model to real (often sheared !) flows, as noted in [18].

6.2. The slow flow regime

Assuming (H1 – H2b – H4), we proceed for the viscous limit of viscoelastic fluids as usual. We specify (H2) as (H2b) : $\boldsymbol{u}_H|_{z=b} = O(\varepsilon)$ and next require $\boldsymbol{T}_{Hz} = O(\varepsilon)$ as above in the inertial case, in addition to (H4) : $\partial_z \boldsymbol{u}_H = O(1)$. Recall also that the flow is necessary slow here ($\boldsymbol{u}_H = O(\varepsilon)$) and one obtains from the momentum balance

$$\frac{1}{\mathrm{Re}}\left((1-\theta)\partial_z \boldsymbol{u}_H + \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}_{Hz}\right) = \boldsymbol{f}_H(z-(b+h)) + O(\varepsilon^2) \tag{84}$$

after using $\boldsymbol{T}_{Hz}|_{z=b+h} = O(\varepsilon^2)$ and $\int_z^{b+h} \operatorname{div}_H(\boldsymbol{T}_{HH} - T_{zz}) = O(\varepsilon^2)$.

Assuming (H3) : Re $\sim \varepsilon^{-1}$ plus (H7) for the sake of simplicity in addition to (H1 – H2b – H4) (and of course De ~ 1, $\theta \sim 1$ as long as nothing different is precised for these nondimensional numbers) leads to a reduced model that is an autonomous system of equations for $(h^0, \boldsymbol{\sigma}^0_{Hz}, \sigma^0_{zz})$

$$\partial_t h^0 + \frac{1}{1-\theta}\operatorname{div}_H\left(\frac{\mathrm{Re}}{6}\boldsymbol{f}_H|h^0|^3 - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}^0_{Hz}\frac{|h^0|^2}{2}\right) = 0 \tag{85a}$$

$$\mathrm{De}\left(\partial_t(h^0\boldsymbol{\sigma}^0_{Hz})\right) = h^0\boldsymbol{\sigma}^0_{Hz} + h^0\frac{\mathrm{De}}{1-\theta}\left(\frac{\mathrm{Re}}{2}\boldsymbol{f}_H - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}^0_{Hz}\right)\sigma^0_{zz}\,, \tag{85b}$$

$$\mathrm{De}\left(\partial_t(h^0\sigma^0_{zz})\right) = h^0(\sigma^0_{zz} - 1)\,, \tag{85c}$$

where the discharge in the continuity equation is computed from (84) and

$$\boldsymbol{u}_H = \frac{1}{1-\theta}\left(\frac{\mathrm{Re}}{2}\boldsymbol{f}_H\,(z-(b+h))^2 - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}^0_{Hz}(z-b)\right) + O(\varepsilon^2)\,, \tag{86}$$

and the longitudinal stress components are obtained by the post-processing

$$\begin{aligned}&\mathrm{De}\left(\partial_t(h^0\boldsymbol{\sigma}^0_{HH})\right) = \boldsymbol{\sigma}^0_{HH} - \boldsymbol{I}\\ &+ h^0\frac{1}{1-\theta}\mathrm{De}\left(\boldsymbol{\sigma}^0_{Hz}\otimes\left(\frac{\mathrm{Re}}{2}\boldsymbol{f}_H - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}^0_{Hz}\right)\left(\frac{\mathrm{Re}}{2}\boldsymbol{f}_H - \theta\frac{1}{\mathrm{De}}\boldsymbol{\sigma}^0_{Hz}\right)\otimes\boldsymbol{\sigma}^0_{Hz}\right)\,.\end{aligned} \tag{87}$$

Assuming (H5a) : $1-\theta \sim \varepsilon$ and (H6a) : $\boldsymbol{\sigma}_{Hz} = O(\varepsilon)$ again requires $\partial_z \boldsymbol{u}_H = O(\varepsilon)$ and cannot be coherent, so we consider (H5a) with (H6c) : De $\sim \varepsilon^{-1}$ only, which leads to $\sigma_{zz} = 1 + O(\varepsilon)$ constant (equal to physical equilibrium) and a reduced model consisting of the limits of (85a) and (85b) as $1/\mathrm{De} \to 0$. (with (87) for post-processing $\boldsymbol{\sigma}^0_{HH}$ only).

Last, assuming (H5b) : $\partial_z \boldsymbol{u}_H = O(\varepsilon)$ and (H6b) : $\theta = O(\varepsilon)$ leads to the same reduced model as the first one above (and is coherent under the more restricitive regime where $O(\varepsilon^2)$ is replaced by $O(\varepsilon^3)$ in (86)), while (H5b) and (H6c) gives the same as the second one above.

All these systems seem new to us: other viscous limits of non-Newtonian viscoelastic fluid models have already been derived, but on assuming different scalings, see e.g. [10, 9, 11] (De $\sim \varepsilon$).

Remark 4 (Nonlinear differential constitutive equations and HWNP). *The most used variations of the UCM model are nonlinear modifications of these differential constitutive equations, for instance the FENE-P model where the extra-stress reads* $\boldsymbol{\tau} = \frac{\theta}{DeRe}\left(\frac{\boldsymbol{\sigma}}{1-\operatorname{tr}\boldsymbol{\sigma}/b} - \boldsymbol{I}\right)$. *The new parameter* $b > 0$ *is such that* $0 \le \operatorname{tr}\boldsymbol{\sigma} \le b$ *(this is preserved by smooth time evolutions of the flow). The conformation tensor* $\boldsymbol{\sigma}$ *is solution to the nonlinear equation*

$$De\left(D_t\boldsymbol{\sigma} - (\nabla\boldsymbol{u})\boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla\boldsymbol{u})^T\right) = \boldsymbol{I} - \frac{\boldsymbol{\sigma}}{1-\operatorname{tr}\boldsymbol{\sigma}/b}\,. \tag{88}$$

One nice feature is that the constraint $0 \le \operatorname{tr}\boldsymbol{\sigma} \le b$ *is believed to alleviate deficiencies of the UCM model (High-Weissenberg-Number Problems or HWNP in short) in the "High-Weissenberg limit" (at least, well-posedness has sometimes been shown for smooth flows, see e.g. [43]). Now, reduced models are then still easily derived as long as one does not use* (H6a) : $\boldsymbol{\sigma}_{Hz} = O(\varepsilon)$. *It suffices to multiply the last term on the right by* $\frac{1}{1-\operatorname{tr}\boldsymbol{\sigma}/b}$, *which is indeed never small,*

- *in* (77b), (77c) *and* (77e) *under* (H1 − H2a − H4 − H3 − H7),
- *in* (82b), (82c) *and* (82e) *under* (H1 − H2a − H4 − H5b − H6b − H7).

On the contrary, since $\frac{1}{1-\operatorname{tr}\boldsymbol{\sigma}/b}$ *can become arbitrary large when* $\operatorname{tr}\boldsymbol{\sigma} \to b$, *this is not only incompatible with* (H6a) : $\boldsymbol{\sigma}_{Hz} = O(\varepsilon)$, *but also requires additional assumptions in the case* (H6c) : $De \sim \varepsilon^{-1}$ *(thus not treated here).*

Another way to avoid HWNP is to assume $De \sim \varepsilon$ *like in e.g. [10, 9, 11] ! Then, one cannot expect strong viscoelastic influences on the flow, of course. Though, this scaling may be enough for some applications, and we would like to mention that it has recently raised interesting new persepctives: a new approach to formal model reduction combining micro and macro scales [47] that is indeed consistent with a Newtonian behaviour in the limit* $De \to 0$.

7. Conclusion

We have defined a mathematical framework that allows to derive *coherent* long-wave thin-layer approximations to the free-surface Navier-Stokes flows of fluids with many various possible rheologies. For each rheology, different reduced models have been derived depending on assumptions about the internal stresses and about the flow, which is assumed driven by gravity on a rugous slowly varying topography. The different models are mainly of two kinds: fast or slow.

Most reduced models derived herein were already known, and the shallow water equations in particular have already proved useful in the numerical simulation of specific situations (dam-breaks for instance). But on the one hand, they do not seem to have been derived in a single unifying framework yet. Moreover, on the other hand, the models for viscoelastic fluids seem to have been much less explored, and some of those derived herein seem new to us. Of course, the question how well the latter actually model real flows is still to be answered. This could be investigated numerically. To that aim, one could follow the same path as in our previous work [18] where we considered the viscoelastic 1D fast flows without friction nor surface tension. See also our recent extension to a 1D case with friction and inclined gravity effects [20].

We hope that the unified framework derived herein will help characterize features essential to long-wave thin-layer flow modelling, and help evaluate the quality of various rheological models applied to geophysical free-surface flows.

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References

[1] F. Abergel and J. L. Bona, *A mathematical theory for viscous, free-surface flows over a perturbed plane*, Arch. Rational Mech. Anal. **118** (1992), no. 1, 71–93. MR 1151927 (92k:35213)

[2] Y. Achdou, O. Pironneau, and F. Valentin, *Effective boundary conditions for laminar flows over periodic rough boundaries*, J. Comp. Phys. **147** (1998), 187–218.

[3] G. Allain, *Small-time existence for the Navier-Stokes equations with a free surface and surface tension*, Free boundary problems: application and theory, Vol. IV (Maubuisson, 1984), Res. Notes in Math., vol. 121, Pitman, Boston, MA, 1985, pp. 355–364. MR 903286 (88h:35093)

[4] Christophe Ancey, *Plasticity and geophysical flows: A review*, Journal of Non-Newtonian Fluid Mechanics **142** (2007), no. 1-3, 4 – 35, In *Viscoplastic fluids: From theory to application.*

[5] Christophe Ancey and Steve Cochard, *The dam-break problem for herschel-bulkley viscoplastic fluids down steep flumes*, Journal of Non-Newtonian Fluid Mechanics **158** (2009), no. 1-3, 18 – 35, Visco-plastic fluids: From theory to application.

[6] H.A. Barnes, J.F. Hutton, and K.F.R.S. Walters, *An introduction to rheology*, 1st ed., Elsevier Science Publisher, Amsterdam, The Netherlands, 1989.

[7] Arnaud Basson and David Gérard-Varet, *Wall laws for fluid flows at a boundary with random roughness*, Comm. Pure Appl. Math. **61** (2008), no. 7, 941–987. MR 2410410 (2009h:76055)

[8] P.Bates, M.Stewart, G.Siggers, C.Smith, J. Hervouet and R.Sellin *Internal and external validation of a two-dimensional finite element code for river flood simulations* Proceedings of the Institution of Civil Engineers: Water Maritime and Energy **130** (1998), 127–141

[9] G. Bayada, L. Chupin, and B. Grec, *Viscoelastic fluids in thin domains: a mathematical proof*, Asymptotic analysis **64** (2009), no. 3, 185–211.

[10] G. Bayada, L. Chupin, and S. Martin, *Viscoelastic fluids in a thin domain*, Quarterly of Applied Mathematics **65** (2007), no. 4, 625–652.

[11] ______, *Viscoelastic Fluids in a Thin Domain: A Mathematical Study for a Non-Newtonian Lubrication Problem*, Mathematical Modeling, Simulation, Visualization and e-Learning (D. Konaté, ed.), Springer, 2010, pp. 315–321.

[12] Guy Bayada and Michèle Chambat, *The transition between the stokes equations and the reynolds equation: A mathematical proof*, Applied Mathematics and Optimization **14** (1986), no. 1, 73–93. MR MR0826853

[13] J. Thomas Beale and Takaaki Nishida, *Large-time behavior of viscous surface waves*, Recent topics in nonlinear PDE, II (Sendai, 1984), North-Holland Math. Stud., vol. 128, North-Holland, Amsterdam, 1985, Lect. Notes Numer. Appl. Anal. (Vol. 8), pp. 1–14. MR 882925 (88f:35121)

[14] D. J. Benney, *Long waves on liquid films*, J. Math. Phys. **45** (1966), 50–155.

[15] R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of polymeric liquids*, vol. 1: Fluid Mechanics, John Wiley & Sons, New York, 1987.

[16] ______, *Dynamics of polymeric liquids*, vol. 2: Kinetic Theory, John Wiley & Sons, New York, 1987.

[17] P. Bonneton and D. Lannes, *Derivation of asymptotic two-dimensional time-dependent equations or surface water wave propagation*, Physics of Fluids **2009** (21), 016601.

[18] François Bouchut and Sébastien Boyaval, *A new model for shallow viscoelastic fluids*, M3AS **23**, no. 8, 1479–1526 (2013).

[19] Francois Bouchut and Michael Westdickenberg, *Gravity driven shallow water models for arbitrary topography*, Commun. Math. Sci. **2** (2004), no. 3, 359–389. MR MR2118849 (2005m:76026)

[20] Sébastien Boyaval *A new model for shallow viscoelastic free-surface flows forced by gravity on rough inclined bottom*, ESAIM Proc. **45** (2014), 108–117.

[21] Sébastien Boyaval, Claude Le Bris, Tony Lelièvre, Yvon Maday, Cuong Nguyen, N. and Tony A. Patera, *Reduced Basis techniques for stochastic problems*, Archives of Computational Methods in Engineering, Springer Netherlands, **17** (2010), 435–454.

[22] Sébastien Boyaval, Tony Lelièvre, and Claude Mangoubi, *Free-energy-dissipative schemes for the Oldroyd-B model*, M2AN Math. Model. Numer. Anal. **43** (2009), no. 3, 523–561. MR 2536248 (2010k:65197)

[23] D. Bresch, E. D. Fernàndez-Nieto, I. R. Ionescu, and P. Vigneaux, *Augmented lagrangian method and compressible visco-plastic flows: Applications to shallow dense avalanches*, New Directions in Mathematical Fluid Mechanics (Giovanni P. Galdi, John G. Heywood, Rolf Rannacher, Andrei V. Fursikov, and Vladislav V. Pukhnachev, eds.), Advances in Mathematical Fluid Mechanics, Birkhäuser Basel, 2010, 10.1007/978-3-0346-0152-8_4, pp. 57–89.

[24] Didier Bresch and Pascal Noble, *Mathematical justification of a shallow water model*, Methods Appl. Anal. **27** (2007), no. 2, 87–118.

[25] A., Caboussat and S., Boyaval and A., Masserey, *On the Modeling and Simulation of Non-Hydrostatic Dam Break Flows*, Computing and Visualization Science, **14** (2013), 401–417

[26] Castro, A. and Lannes, D. *Fully nonlinear long-wave models in the presence of vorticity*, Journal of Fluid Mechanics **759** (2014), 642–675

[27] Chanson, H. *The Hydraulics of Open Channel Flow: An Introduction.* (Second Edition) Butterworth-Heinemann, 2004

[28] Laurent Chupin, *The FENE viscoelastic model and thin film flows*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 17–18, 1041–1046.

[29] ———, *The FENE viscoelastic model and thin film flows*, Methods and Applications of Analysis **16** (2009), no. 1, 217–262.

[30] M. Cruchaga, L. Battaglia, M. Storti and J. D'Elía *Numerical Modeling and Experimental Validation of Free Surface Flow Problems*, Archives of Computational Methods in Engineering, **14** (2014), 1–31.

[31] R. V. Craster and O. K. Matar, *Dynamics and stability of thin liquid films*, Rev. Mod. Phys. **81** (2009), 1131–1198.

[32] G. Duvaut and J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, 1972.

[33] E. D. Fernández-Nieto, P. Noble, and J.-P. Vila, *Shallow water equations for non newtonian fluids*, Journal of Non-Newtonian Fluid Mechanics **165** (2010), no. 13–14, 712–732.

[34] David Gérard-Varet, *The navier wall law at a boundary with random roughness*, Communications in Mathematical Physics **286** (2009), 81–110 (English).

[35] Jean-Frédéric Gerbeau and Benoît Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water ; numerical validation*, Discrete and continuous dynamical system Series B **1** (2001), no. 1, 89–102.

[36] Jean-Michel Hervouet, *Hydrodynamics of free surface flows. Modelling with the finite element method.*, Wiley & Sons, 2007.

[37] Pierre Jop, Yoël Forterre, and Olivier Pouliquen, *A constitutive law for dense granular flows*, Nature **441** (2006), no. 7094, 727–730.

[38] Lamb, *Hydrodynamics*, Cambridge University Press, 1975.

[39] P. Noble J.-P. Vila M. Boutounet, L. Chupin, *Shallow water equations for newtonian fluids over arbitrary topographies*, Comm Math Sci **6** (2008), no. 1, 29–55.

[40] F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects*, European Journal of Mechanics-B/Fluids **26** (2007), no. 1, 49–63.

[41] V. Maronnier, M. Picasso and J. Rappaz, *Numerical simulation of three-dimensional free surface flows*, International Journal for Numerical Methods in Fluids **42** (2003), 697–716

[42] J. D., Martin and W. J. Moyce, *Part IV. An experimental study of the collapse of liquid columns on a rigid horizontal plane*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, **244** (1952), 312–324

[43] N. Masmoudi, *Global existence of weak solutions to macroscopic models of polymeric flows*, Journal de Mathématiques Pures et Appliquées **96** (2011), no. 5, 502–520.

[44] S. Kalliadasis, C. Bielarz and G. M. Homsy, *Steady free-surface thin film flows over topography* Physics of Fluids **12** (2000), 1889–1898

[45] Hervé Le Meur, *Well-posedness of surface wave equations above a viscoelastic fluid*, Journal of Mathematical Fluid Mechanics **13** (2011), no. 4, 481–514.

[46] H. B. H. Mohamed and B. D. Reddy, *Some properties of models for generalized Oldroyd-B fluids*, Internat. J. Engrg. Sci. **48** (2010), no. 11, 1470–1480, Special Issue in Honor of K.R. Rajagopal. MR 2760996

[47] G. Narbona-Reina and D. Bresch, *On a shallow water model for non-newtonian fluids*, Numerical Mathematics and Advanced Applications 2009 (Gunilla Kreiss, Per Lötstedt, Axel Møalqvist, and Maya Neytcheva, eds.), Springer Berlin Heidelberg, 2010, 10.1007/978-3-642-11795-4 74, pp. 693–701.

[48] Alexander Oron, Stephen H. Davis, and S. George Bankoff, *Long-scale evolution of thin liquid films*, Rev. Mod. Phys. **69** (1997), no. 3, 931–980.

[49] M. Renardy, *Mathematical analysis of viscoelastic flows*, CBMS-NSF Conference Series in Applied Mathematics, vol. 73, SIAM, 2000.

[50] M. Renardy, *The mathematics of myth: Yield stress behavior as a limit of non-monotone constitutive theories*, Journal of Non-Newtonian Fluid Mechanics **165** (2010), 519–526.

[51] C. Ruyer-Quil and P. Manneville, *Improved modeling of flows down inclined planes*, Eur. Phys. J. B **15** (2000), no. 2, 357–369.

[52] A.J.C. de Saint-Venant, *Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit*, C. R. Acad. Sc. Paris. **73** (1871), 147–154.

[53] P. Saramito, *A new elastoviscoplastic model based on the Herschel-Bulkley viscoplasticity*, J. Non Newtonian Fluid Mech. **158** (2009), no. 1–3, 154–161.

[54] Yoshihiro Shibata and Senjo Shimizu, *On a free boundary problem for the Navier-Stokes equations*, Differential Integral Equations **20** (2007), no. 3, 241–276. MR 2293985 (2008a:35211)

[55] J. J. Stoker, *Water Waves: The Mathematical Theory with Applications* (1957), Interscience, London.

[56] P. Viollet, J. Chabard, P. Esposito and D. Laurence, *Mcanique des fluides applique: coulements incompressibles dans les circuits, canaux et rivires, autour des structures et dans l'environnement* (1998) Presses de l'Ecole nationale des ponts et chausses.

[57] Peter Wapperom and Martien A. Hulsen, *Thermodynamics of viscoelastic fluids: the temperature equation*, J. Rheol. **42** (1998), no. 5, 999–1019.

[58] Whitham, G. B. *Linear and Nonlinear Waves* (1999), John Wiley & Sons, Inc., Hoboken, NJ, USA.