Optimal timing of CO_2 mitigation policies for a cost-effectiveness model.

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Abstract

This paper deals with the regulation of greenhouse gases emissions related to climate change. We consider a stylized climate-economy sequential model and use a cost-effectiveness approach. The analytical study is based on dynamic programming method. It provides both a tolerable ceiling of concentration and, under simple conditions involving the marginal abatement cost and emission functions, optimal and effective abatement rates. In particular, we prove how the cost effective abatement rate increases with time. Through the optimal time to act function, we examine in detail the role played by greenhouse gases absorption, growth and discount rates. We also analyze the paths from an intergenerational equity perspective. Numerical examples illustrate the general statements.

1 Introduction

Climate change has now emerged as one of the most important issues facing the international community as it could durably threaten productive activities, human settlements and environmental amenities and act as a supplementary barrier to development in many regions of the world. Over the past decade, many efforts have been directed toward evaluating policies to control the atmospheric accumulation of greenhouse gases (GHG), especially carbon dioxide (CO_2) which will be responsible for the main part of the additional

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anthropogenic atmospheric forcing during the 21st century. Due to the lack of reliable information on the magnitude of climate change damages, particular attention has been paid to the stabilization of GHG concentrations. In spite of many analyses, surveyed for instance in IPCC (2001a), considerable debates about the timing and magnitude of GHG emissions mitigation in this context still take place owing to significant sources of uncertainty applying to socio-economic features (technology development, capital stock, societies inertias and discounting) and environmental determinants (carbon cycle). These discussions emphasize the necessity to deal explicitly with environmental, economic and technological imprecisions, uncertainties and ambiguity in view to draw reliable operational conclusions to cope with climate change following a precautionary approach. Furthermore, intergenerational equity concerns related to sustainability preoccupations play an important role in the mitigation policy and it is still unclear whether the choice of the discount rate is enough to yield relevant answers in this topic Heal (1998).

Numerous studies have challenged these issues by using different approaches regarding decision framework and model complexity. At this stage, we can distinguish several classes of models: simulation, cost-benefit, cost-effectiveness and viability models. Simulation or policy evaluation models, such as IMAGE Alcamo (1994), produce detailed pictures of climate change impacts following exogenous socio-economic scenarios (including prices, technological, demographic and life-style projections). Here, uncertainty is dealt with using different scenarios. This class of models allows for a fairly detailed view of the system, but may become complex and does not easily allow for an analytic comparison of policies. To overcome these difficulties, control or decision frameworks are used, involving less complex and more compact or stylized models. In particular, cost-benefit analysis Farzin and Tahvonen (1996); Goulder and Mathai (2000); Kolstad (1996b); Manne et al. (1995); Nordhaus (1994) refers to an inter-temporal criteria of discounted net benefits (*i.e.* entailed costs and avoided damages) which are expressed in a homogeneous metric, usually as a ratio of gross world product. In this context, the policy is associated to an optimal dynamic decision. In Tolerable Windows Approaches Petschel-Held et al. (1999) or viability and target concerns Hourcade and Chapuis (1995), the performance of the sequential decisions is evaluated through the respect of given constraints that stands for the admissibility, the safety, or by extension the effectiveness of the decision strategy. Thus a set of acceptable states and policies respecting some constraints are computed or approximated. The use of constraints is a way to avoid the detailed specification of a damage function and to take into account the lack of knowledge in this area Ambrosi et al. (2003). The advantage of these geometric approaches is to provide flexibility, some room of maneuver for decisions and to stress on irreversibility and dangerous situations. It is possible to mix these two approaches within a cost-effectiveness framework where an inter-temporal actualized cost is minimized given some admissible environmental constraints Manne et al. (1995); Goulder and Mathai (2000); Ha-Duong et al. (1997). In most of optimal approaches, uncertainty is taken into account through stochastic and probabilistic assumptions. Hence the decision refers to the maximization of the expected criteria Gjerde et al. (1999); Kolstad (1996a); Ulph and Ulph (1997). Within this context, the role of information and learning is important.

The present paper proposes and analyzes a dynamic control problem in a cost-effectiveness framework. Our work is in the spirit of Farzin and Tahvonen (1996), Goulder and Mathai (2000) or Peck and Wan (1996) in the sense that it focuses on analytical results for mitigation decisions. Nevertheless, instead of deriving results from the so-called maximum principle, we use dynamic programming and backward Bellman method. One interest of such an approach is to reveal feedback policies (control depending on the state) which are known to display important properties of adaptability. It also allows to provide explicit qualitative, quantitative and sensitivity statements. Of course this analytical perspective implies to impose some simplifying assumptions, but the interest is clearly to point out some mechanisms at stake in physical, economic and decision processes.

The model represents the interactions between economic growth and the progressive build-up of GHG concentrations (following a linear carbon cycle). Here the influence of the economy on the environment is captured through emission and abatement cost functions, both depending on the gross world product (GWP). We do not specify the emission function simply requiring a positive elasticity with respect to the economic activity level. For sake of simplicity, we assume the cost to be proportional with respect to abatement rates (*i.e.* linear, unlike all previous studies) still we do not specify the form of the marginal abatement cost function (but a general hypothesis of decrease with time). The main questions that we address in this paper are:

- What is the behaviour of optimal and effective sequential abatements?
- What is the associated mitigation timing?
- Does the effective policy exhibit intergenerational equity?
- What is the sensitivity of the quantitative results?

The paper is organized as follows. Section 2 lays out the dynamic model, including the discrete time dynamics, the target, the inter-temporal criteria along with the optimization problem. Section 3 presents the analytical results focusing on the tolerable ceiling, the optimal feedback policy and the study of the “optimal time to act” value. Section 4 provides some numerical illustrations. The final section concludes and presents future developments of research. To restrict the mathematical content in the core of the text, the proofs of the formal propositions are expounded in the appendix.

2 The model

The model is a discrete time one and thus refers to sequential decision. The state of climate-economy system at time $t \geq 0$ is described by two variables, namely some aggregated economic production level such as gross world product GWP denoted by Q_t and the atmospheric GHG concentration level denoted by M_t. The decision variable related to mitigation policy is the emission abatement rate denoted by a_t. The goal of the policy makers is to minimize inter-temporal discounted abatement costs while respecting a maximal sustainable GHG concentration threshold at the final time horizon. Thus we face a cost-effectiveness problem.

2.1 Discrete dynamics of the system

The description of the carbon cycle is similar to Nordhaus (1994)

$$M_{t+1} = M_t + E(Q_t)(1 - a_t) - \delta(M_t - M_\infty) \tag{1}$$

where

- the function $E(Q)$ stands for the emissions of GHG resulting from the economic production Q in a“Business As Usual” (BAU) scenario and accumulating in the atmosphere.
- the abatement rate a_t corresponds to the applied reduction of GHG emissions level ($0 \leq a_t \leq 1$)
- the parameter δ stands for the natural rate of removal of atmospheric CO_2 to unspecified sinks ($0 \leq \delta < 1$).

It can be noticed that carbon cycle dynamics can be reformulated as

$$M_{t+1} - M_\infty = (1 - \delta)\,(M_t - M_\infty) + E(Q_t)(1 - a_t) \tag{2}$$

thus representing the anthropogenic perturbation of a natural system from a preindustrial equilibrium atmospheric concentration M_∞. Thus δ accounts for the inertia of natural system. Two polar cases are worth being pointed out: when $\delta = 1$, carbon cycle inertia is nil and therefore GHG emissions induce a flow externality; on the contrary, when $\delta = 0$, stock externality reaches a maximum and GHG accumulation is irreversible[1].

We do not provide a specific form for the baseline emissions function $E(.)$ allowing for non-linear emission mechanisms. The emissions depend on production Q because it is clear that growth is a major determinant on energy demand Manne et al. (1995). We only assume that BAU emissions increase[2] with production Q namely when E is regular enough

$$\frac{\partial E(Q)}{\partial Q} > 0. \tag{3}$$

Combined with a global economic growth assumption, this is equivalent to a rising emissions baseline.

The global economy dynamics is represented by an autonomous rate of growth $g \geq 0$ for the aggregated production level Q_t related to gross world product:

$$Q_{t+1} = (1+g)Q_t. \tag{4}$$

This dynamics means that the economy is not directly affected by abatement policies and costs. Of course, this is a restrictive assumption but it is commonly used in modeling for GHG reduction policies as in Ambrosi et al. (2003) or Ha-Duong et al. (1997). However, one might consider this GWP dynamics (4) as a relevant global economic target or a constraint related to sustainable development.

[1]The removal rate, δ, is a most uncertain parameter and some recent results indicate that it may decrease due to the combined effect of high GHG accumulation and global warming (the so-called climate-carbon cycle feedback in IPCC (2001b)) and ongoing deforestation trends Gitz and Ciais (2003).

[2]We do not need to assume that emissions E increase more slowly than production, a frequent hypothesis, in the sense that

$$\frac{\partial^2 E(Q)}{\partial^2 Q} < 0.$$

2.2 The cost-effectiveness criteria

2.2.1 The effectiveness setting

We consider a physical or environmental requirement through the limitation of concentrations of GHG below a tolerable threshold $M^\flat$ at a specified date $T > 0$:

$$M_T \leq M^\flat. \tag{5}$$

Observe that we can equivalently write a damage function with an indicatric extended function

$$D(M) = \begin{cases} 0 & \text{if} \quad M - M^\flat \leq 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

2.2.2 The abatement costs

For sake of simplicity, we assume that the abatement costs $C(a, Q)$ are of linear form with respect to abatement rate a in the sense that

$$C(a, Q) = c(Q)a. \tag{7}$$

We do not specify the marginal cost function allowing again for non linear processes. We just assume that the abatement cost $C(a, Q)$ increases with a which implies

$$\frac{\partial C(a, Q)}{\partial a} = c(Q) > 0. \tag{8}$$

Furthermore, following for instance Goulder and Mathai (2000) or Peck and Wan (1996), we assume that growth lowers marginal abatement costs[3]. This means that the availability and costs of technologies for fuel switching improve with growth. Thus if the marginal abatement cost $c(.)$ is regular enough, it is decreasing with production in the sense

$$\frac{\partial^2 C(a, Q)}{\partial Q \partial a} = \frac{\partial c(Q)}{\partial Q} \leq 0. \tag{9}$$

As a result, the costs of reducing a ton of carbon decline.

[3] We could reason similarly with abatement level $A = aE(Q)$. In this case indeed, we could define the reduction cost

$$\widetilde{C}(A, Q) = C\left(\frac{A}{E(Q)}, Q\right) = A\frac{c(Q)}{E(Q)}.$$

Thus, since emissions $E(.)$ increases with production Q, the marginal costs $\widetilde{C}_A(Q) = \frac{c(Q)}{E(Q)}$ decreases more sharply.

2.2.3 The optimisation problem

The cost effectiveness problem faced by the social planner is an optimisation problem under constraints. It consists in minimizing the discounted intertemporal abatement cost

$$\sum_{t=0}^{T} \rho^t C(a_t, Q_t) \tag{10}$$

while reaching the concentration tolerable window $M_T \leq M^\flat$. The parameter ρ stands for the discount factor of the considered period. Therefore, the problem can be written in a value function form as follows:

$$V(0, M_0, Q_0) \quad = \quad \min_{a_0, a_1, .., a_{T-1}} \left(\sum_{t=0}^{T} \rho^t C(a_t, Q_t) + \rho^T D(M_T) \right), \tag{11}$$

under the dynamics constraints (1) and (4). We denote by $a_0^*, a_1^*, .., a_{T-1}^*$ an optimal solution of the previous problem whenever it exists.

3 Optimal and effective abatements

Using backward dynamic programming approach, we compute explicitly the optimal and feasible solutions of the cost-effectiveness problem. At this stage, let us mention that the proofs are not obvious and require the use of generalized gradient Rockafellar and Wets (1998). Indeed, the value function and feedback controls display some non smooth shapes because of kink solutions and active constraints.

3.1 A viability ceiling

We first provide an existence or effectiveness result whose proof is given in the appendix A.3 and derived from proposition (A.1). We need to introduce the following maximal concentration values

$$M_t^\sharp = (M^\flat - M_\infty)(1 - \delta)^{t-T} + M_\infty. \tag{12}$$

These induced thresholds $M_t^\sharp$ play the role of an evolving tolerable window and account for irreversibility constraints.

Proposition 3.1 *An optimal cost-effective policy exists if and only if the initial concentration M_0 is smaller than $M_0^\sharp$. In that case, the whole policy $a_0, a_1, .., a_{T-1}$ is effective if and only if associated concentrations $M(t)$ remain lower than $M_t^\sharp$.*

Let us observe that we always have $M_T^\sharp = M^\flat$ which means that the terminal tolerable concentration is $M^\flat$ as expected. We note also that, whenever the natural removal occurs ($\delta > 0$), safety thresholds $M_t^\sharp$ are strictly larger than terminal target $M^\flat$, which allows for exceeding the target during time. *A contrario*, whenever the natural removal term disappears ($\delta = 0$), the induced safety thresholds coincide with final $M^\flat$ along the whole time sequence and the effective mitigation policy imposes to stay below the CO_2 target at every period.

3.2 Optimal feedback

Now, under the previous existence and effectiveness assumption, we obtain the optimal policy in terms of a feedback depending on the current environmental-economic state (M, Q) of the system. The proof based on Bellman dynamic programming principle is given in the appendix.

Proposition 3.2 *Consider a tolerable initial situation $M_0 \leq M_0^\sharp$. If assumptions for emissions and cost functions (3) and (9) hold true, then the optimal effective mitigation policy is defined by the feedback abatement*

$$a^*(t, M, Q) = \max\left(0, \frac{(1-\delta)(M - M_t^\sharp) + E(Q)}{E(Q)}\right).$$

Let us point out that the abatement $a^*(t, M, Q)$ reduces to zero when condition $(1-\delta)(M - M_t^\sharp) + E(Q)$ is negative which corresponds to the case where the violation of the tolerable threshold $M_t^\sharp$ is not at stake even with Business As Usual emissions. We also emphasize that the case of total abatement where $a^*(t, M, Q) = 1$ occurs when current concentration M coincides with maximal tolerable concentration $M_t^\sharp$.

Moreover, let us mention that it is well-known that feedback and closed-loop (depending on the state) decisions or controls are better than open-loop ones because of their adaptive and stability properties. This means that applying such decision rules yields to relevant states even with errors or perturbations occurring along time since the feedback decisions take into account the current state of the system. In the present context, this implies that some events modifying the concentration or the GWP in an unforeseen way could be compensated and assimilated by the abatement decisions and generate a relevant evolution.

3.3 Increasing abatement rates

Again we consider the previous existence and effectiveness assumptions. Using the optimal feedback abatement above, we obtain the following monotonicity result whose proof is given in the appendix A.5. It favors the all in the last minute option in the sense that the reduction of emissions is more intensive at the end of period than at the beginning.

Proposition 3.3 *Consider a tolerable situation $M_0 \leq M_0^\sharp$. If assumptions for emissions and cost functions (3) and (9) holds true, then the optimal mitigation policy sequence a_t^* is increasing with time in the sense that*

$$a_0^* \leq a_1^* \leq .. \leq a_{T-1}^*.$$

At this stage, let us point out that the previous qualitative results do not depend neither on the discount factor $\rho \leq 1$, nor on the growth rate $g \geq 0$, nor on the specific form of the emission and marginal abatement cost functions. This emphasizes the generality of the assertions. In other words, only a change on the described behaviour of emission function or the use of a non linear cost function could justify another abatement decision profile on the ground of this simple optimality model.

3.4 Optimal time to act

However, the timing of action remains a relevant question. Indeed, first optimal abatement rates are zero while the last ones turns out to be extensive. But the date when to jump from a situation of no abatement into a situation of abatement is a key issue. The optimal time to act OTTA(.) function allows to cope with this kind of concern. It is related to the first strictly positive optimal abatement along time:

$$\mathrm{OTTA}(M_0, Q_0) = \inf(t \geq 0 \mid a_t^* > 0).$$

It turns out that the OTTA is closely connected to both the Business As Usual (BAU) trajectories and tolerable ceiling $M_t^\sharp$. We define the Baseline or Business As Usual trajectory M^{bau} as the solution of (1) starting at initial concentration M_0 without abatement *i.e.* with $a(t) = 0$ for any time t:

$$M_{t+1}^{\mathrm{bau}} = M_t^{\mathrm{bau}} + E(Q_t) - \delta(M_t^{\mathrm{bau}} - M_\infty). \qquad (13)$$

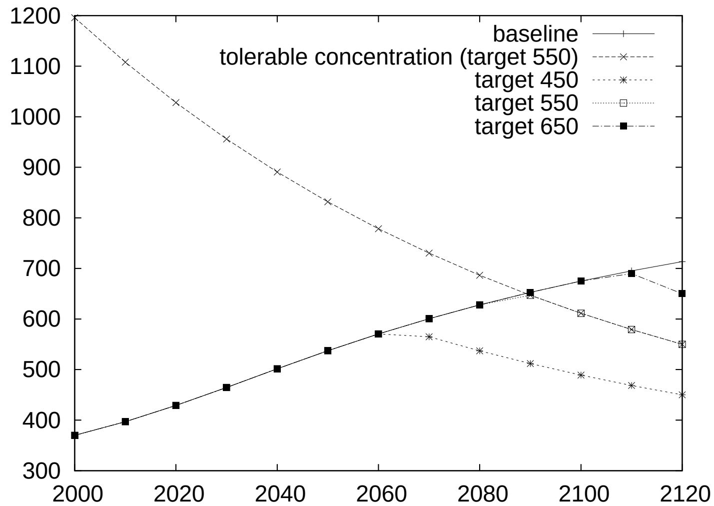


Figure 1: Cost-effective concentrations trajectories M_t^* over the time window [2000, 2120]. The optimal concentrations are plotted for different concentration targets. The baseline concentrations, M_t^{bau}, and the tolerable concentration for a 550 ppm target, $M_t^\sharp$, are also shown.

The proposition (3.4) below, whose proof is given in the appendix A.6, characterizes the optimal time to act through the time when the baseline trajectory M_t^{bau} crosses the tolerable ceiling $M_t^\sharp$.

Proposition 3.4 *The optimal time to act function is given by*

$$\text{OTTA}(M_0, Q_0) = \max\Big(t \geq 0 \mid M_t^{\text{bau}} \leq M_t^\sharp\Big).$$

3.5 Sensitivity concerns

We concentrate the sensitivity analysis with respect to the parameters of the model on optimal time to act function. From previous proposition (3.4), the sensitivity of OTTA is deduced from the variations of M^{bau} and $M^\sharp$. We already know the explicit formulation (12) of the viability ceiling $M^\sharp$. We now expand the expression of the baseline concentrations (13). A simple recursion on carbon cycle leads to:

$$M_t^{\text{bau}} = (1-\delta)^t(M_0 - M_\infty) + M_\infty + \sum_{s=0}^{t}(1-\delta)^{t-s}E(Q_0(1+g)^s).$$

Therefore we derive the following table of sensitivity:

	δ	M_∞	g	T	$M^\flat$	ρ
M^{bau}	-	+	+	0	0	0
$M^\sharp$	+	-	0	+	+	0

Consequently, using previous proposition (3.4), we obtain the detailed sensitivity analysis of OTTA.

Corollary 3.5

(i) OTTA(.) *does not depend on the marginal cost function* $c(Q)$ *and the discount factor* ρ*.*

(ii) OTTA(.) *increases with respect to* δ*,* T *and* $M^\flat$*.*

(iii) OTTA(.) *decreases with respect to* M_∞ *and* g*.*

Most of these results are not amazing and confirm the intuition. Not surprisingly, more intensity of absorption mechanisms provides more flexibility in the prevention action. "When flexibility" is also achieved through the relaxation of the effectiveness target both in time and concentration. Symmetrically one clear result is that economic growth reduces the flexibility of abatement decisions. More surprising is the fact that the discount rate does not affect the mitigation timing. Although one simple explanation of this result is the linear form of the abatement cost function with respect to abatement, this statement emphasizes that the discount context is not always the most stringent determinant for the analysis.

3.6 Mitigation costs and equity concerns

The intergenerational equity issue is important as long as sustainability is at stake. Since production growth is assumed to be exogenous in the model, we capture these intergenerational concerns though the abatement profile $C(a_t, Q_t)$. Since the optimal reductions a_t^* are zero before OTTA, we obviously deduce that optimal costs $C(a_t^*, Q_t)$ are also zero for any generations before OTTA. Furthermore, since abatement policies are extensive (equal to one) for almost every time following OTTA, a clear jump of abatement costs occurs and consequently the optimal mitigation policies do not yield any intergenerational equity. Let us emphasize that this behavior is not related to the discount rate. Moreover, from the declining form of the marginal abatement cost function $c(Q)$, it appears that the cost sequence exhibits a decreasing pattern from OTTA to horizon term T. Consequently, as illustrated by figure 2, there is a sharp "peak" for the OTTA generation.

Let us point out that, in this perspective, the generations older than OTTA could be the "losers" for two reasons: they support the costs of mitigation while probably facing the damages to come.

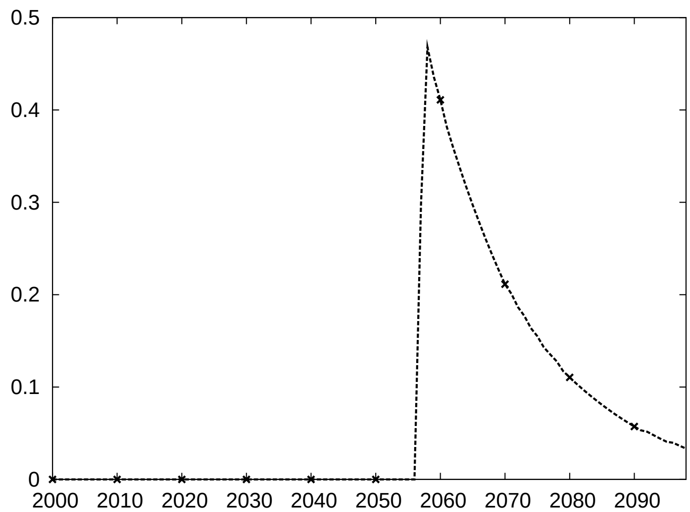


Figure 2: Cost path $\frac{C(a_t^*,Q_t)}{Q_t}$ displays intergenerational inequity. A "peak" appears for generations OTTA.

4 Numerical illustrations

To illustrate the previous analytical results, we now turn to numerical computations performed for specific functional forms and parameter values. In particular, emissions follow a prospective scenario. It is shown that, in the theoretical case relying on a linear marginal abatement costs, as expected

1. the optimal abatement is increasing with respect to time,
2. a quick jump from 0 to 100% reduction rate occurs during time in the optimal abatement policy.

To test the robustness of these results, we relax two of the underlying hypothesis:

1. different non linear increasing marginal abatement cost functions are used;

parameter	η	ξ	α	μ
value	100	-0.03	19.2	0.03

Table 1: Parameters of the abatement cost functions

2. a concentration ceiling $M^\flat$ acting over the whole period, not just the terminal time T.

4.1 Numerical context

For these illustrations, baseline scenarios, the optimal paths and the tolerable concentrations are computed for different values and functional forms inspired from calibrations of Nordhaus (1994) and from the scenario A1M of AIM (1997) in Nakićenović and Swart (2000).

The model is solved at 1-year intervals with 1990 as the initial year ($t = 0$) and 2100 as the target date ($t = T$). Initial condition M_0 is set according to Keeling and Whorf (2002) and Q_0 is set according to AIM (1997)

$$M_0 = 354 \text{ ppmv}, \ Q_0 = 20.9 \text{ T US\$},$$

while preindustrial level is fixed to $M_\infty = 274$ ppmv.

The carbon cycle formulation and parameters are taken from Nordhaus (1994), $\delta = 1/120$. In the dynamic equation (1) the function $E(.)$ stands for the emissions reaching the atmosphere, thus we introduce $E^p(.)$, the emissions released by the production process and we have:

$$E(Q) = \beta E^p(Q) \ (\beta = 0.64).$$

$E^p(.)$ is taken from AIM, points for each year are obtained with a linear interpolation. After 2060 the emissions are set constant at the 2050 level.

The concentration ceiling level is set to $M^\flat = 450$ ppm.

The cost functions $C(a, Q)$ tested have the following multiplicative form:

$$C(a, Q) = P(Q)L(a).$$

The marginal cost function is

$$\frac{dC(a, Q)}{da} = P(Q)l(a)$$

The following functions $l(a)$ are distinguished with the parameters summarized in table 1:

- theoretical case of constant marginal cost

$$l(a) = \eta;$$

- a sigmoid marginal cost function with a backstop technology, *i.e.* a constant marginal cost for high abatement rates (function S):

$$l(a) = \eta + \frac{\xi - \eta}{1 - \frac{\xi}{\eta}\exp(\alpha a)}.$$

  η is the price of the backstop technology, in the sense that $l(1) \approx \eta$. With the value of α used, the backstop price is around η for $a = 0.8$.

- a linearly increasing marginal cost function:

$$l(a) = \eta a;$$

The function $P(Q)$ is assumed to be related to the BAU emissions $E^p(Q)$ as in Ambrosi et al. (2003):

$$P(Q) = E^p(Q)\left(\frac{Q}{Q_0}\right)^{-\mu}$$

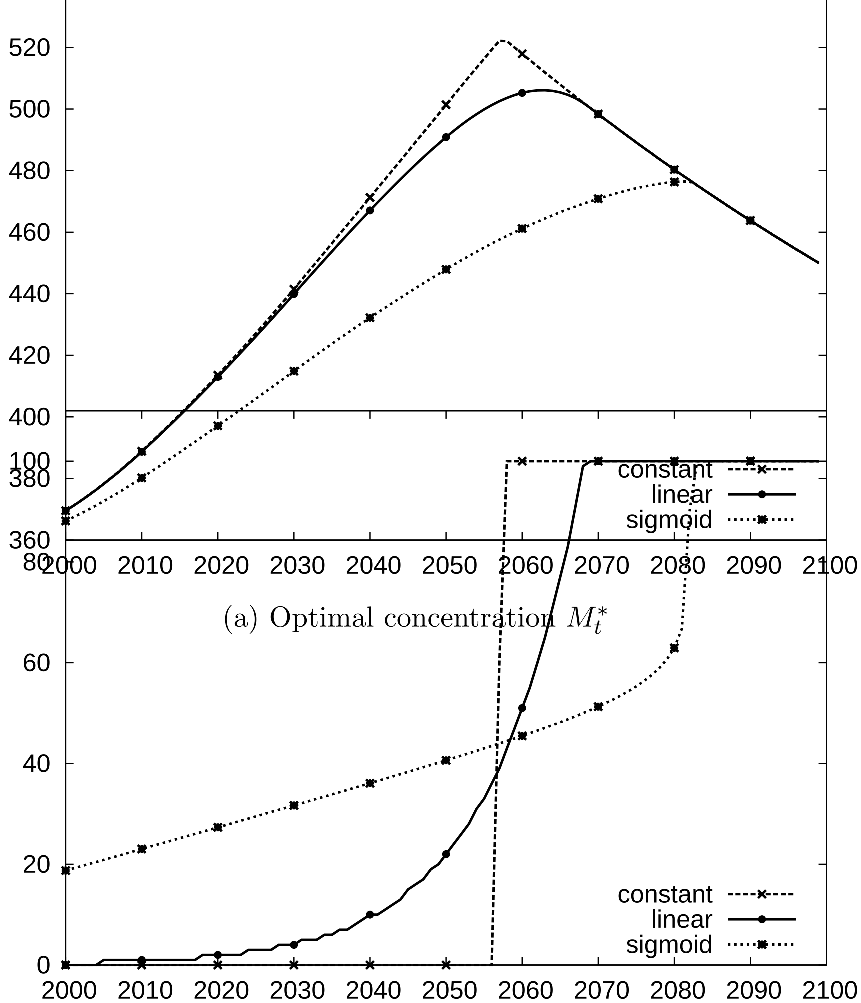
where $\mu > 0$ may rely on technical progress rate. The parameter values are chosen such that the backstop for the sigmoid functions is at 1000$ per ton of carbon, and cost of a full abatement ($a = 1$) is about 550$ per ton of carbon, in the first year.

The proposed cost functions are consistent with the evaluations found in the literature Keith et al. (2005); IPCC (2001a) with an optimistic assumption regarding technical progress.

4.2 Simulation results

Increasing abatement: As displayed by Figure 3(b), it turns out that with all the increasing positive marginal cost function $l(.)$ tested, optimal abatement rate a_t^* remains increasing with respect to time. This suggests the following conjecture that could be studied mathematically extending the results for the linear marginal cost.

Conjecture 1 *If marginal cost function is increasing, optimal abatement is increasing along the optimal trajectory.*



(a) Optimal concentration M_t^*

(b) Optimal abatement rate a_t^* (%)

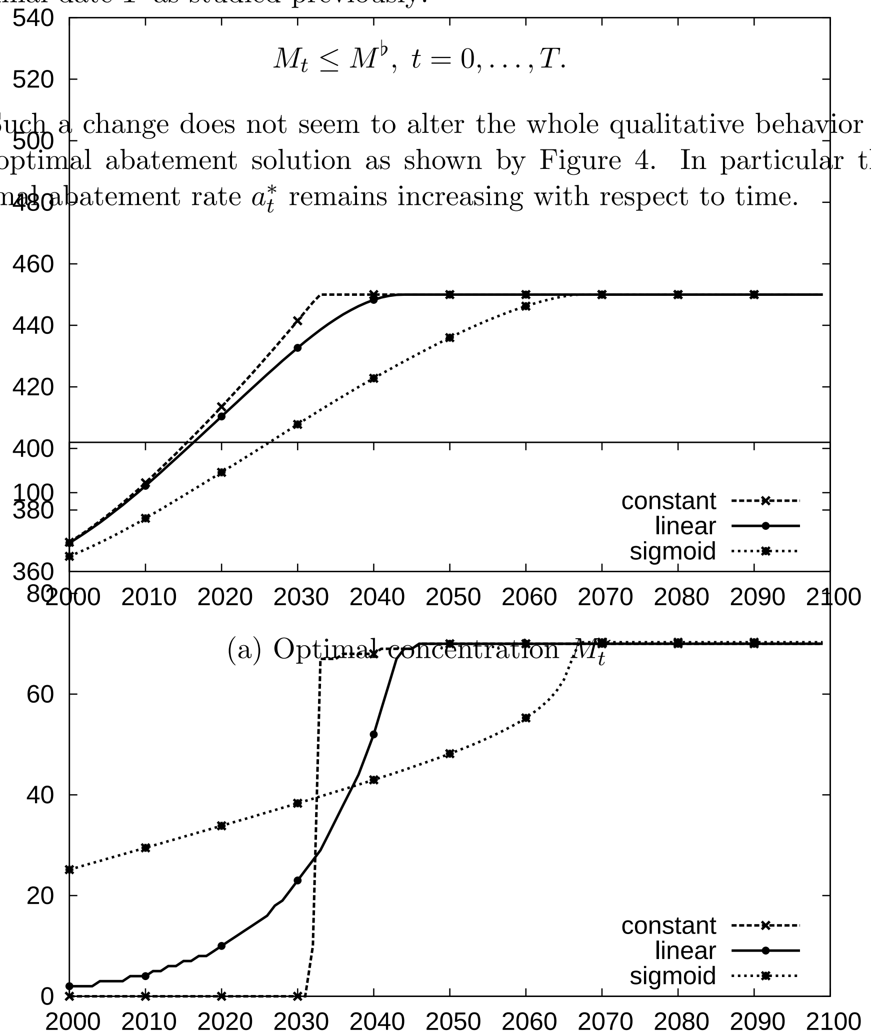
Figure 3: Optimal abatement rate a_t^* and concentration M_t^* over the period $[0, T]$ for distinct marginal cost functions: constant (theoretical case), linear, sigmoid.

Jump and intergenerational equity: We now turn to the presence of a jump in reduced emissions a_t^* jeopardizing the intergenerational equity of mitigation costs. Here the picture seems to be more complicated to extend the linear result. The numerical results captured by Figure 3(b) suggests that such a jump (at least a strong change of slope) appear when there is a taste of constant marginal cost in the marginal function in particular when the function saturates at a constant level. Hence, the jump occurs in the case of constant and sigmoid marginal cost functions while this jump effect vanishes for the linear case.

Concentration constraint from the beginning: We also examined the situation when the CO_2 constraint holds from the beginning not just at the terminal date T as studied previously:

$$M_t \le M^\flat, \ t = 0, \dots, T.$$

Such a change does not seem to alter the whole qualitative behavior of the optimal abatement solution as shown by Figure 4. In particular the optimal abatement rate a_t^* remains increasing with respect to time.

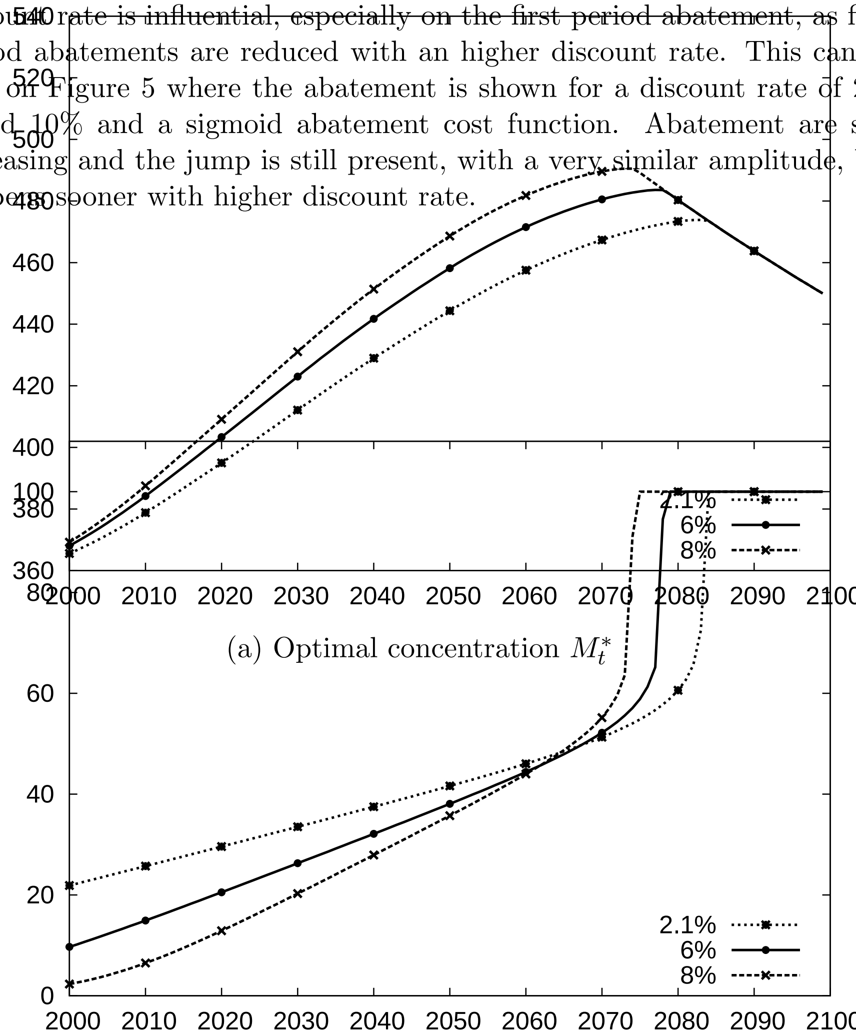


(a) Optimal concentration M_t^*

(b) Optimal abatement rate a_t^* (%)

Figure 4: Optimal abatement rate a_t^* and concentration M_t^* over the period $[0, T]$ with a ceiling concentration $M^\flat = 450$ ppm over the whole time period for distinct marginal cost functions: constant (theoretical case), linear, sigmoid.

Discount rate influence: In the case of a non-linear cost function, the discount rate is influential, especially on the first period abatement, as first period abatements are reduced with an higher discount rate. This can be seen on Figure 5 where the abatement is shown for a discount rate of 2.1, 6 and 10% and a sigmoid abatement cost function. Abatement are still increasing and the jump is still present, with a very similar amplitude, but happens sooner with higher discount rate.



(a) Optimal concentration M_t^*

(b) Optimal abatement rate a_t^* (%)

Figure 5: Optimal abatement rate a_t^* and concentration M_t^* over the period $[0, T]$ with a sigmoid cost function and distinct discount rates: 2.1, 6 and 10%.

Therefore, these experiments indicate how the analytical results found in the context of constant marginal cost may be, at least partially, generalized.

5 Conclusion and perspectives

In this paper, a cost-effectiveness model provides results for greenhouse gases mitigation policies. A tolerable ceiling of concentration is computed together with optimal and effective abatement rates. One main originality of the paper is to shed an analytical and mathematical light on mitigation profile. Bellman principle and backward dynamic programming method allow to compute effective and optimal feedback policies with interesting adaptive properties. The analytical results enable also to deduce explicit sensitivity analyses. Numerical simulations are also given, reinforcing the intuition and understanding of the results. They suggest extensions of the general mathematical results obtained in the linear case.

Although illustrative in nature, the study highlights some important considerations for decision making. First, it turns out that both declining marginal abatement cost and increasing emissions are sufficient conditions for the abatement rates to exhibit an increasing intensity with respect to time. This means that there are strong reasons for preferring emissions strategies involving modest reductions in the near term followed by sharper reductions later on. Second, the study of the optimal time to act function reveals the weak influence of marginal cost function and discount rate on the policy. Furthermore, this optimal time to act function also shows how absorption and removal mechanisms in the carbon cycle favor the flexibility of mitigation decisions whereas the growth intensity seems to reduce it. We also show that the optimal abatement cost profile does not seem fair regarding intergenerational equity.

Nevertheless a lot of work remains to do. One main restriction of the work is the linear behavior of the cost function with respect to abatement rate. This is part of future research to relax this assumption of "first order approximation " and to exhibit more general monotonicity results suggested by our numerical works. Moreover the influence of technology acting on growth, baseline emissions and marginal costs is crucial in the analysis and the extension of the model in this direction is a challenging goal.

The sensitivity analysis shows clear monotone and non local dependencies with respect to the parameters and we might derive some robust decision processes within a radical uncertainty context. However, we are convinced that stochastic control is a relevant framework to overtake uncertainty concerns. More specifically, in this context, much attention has to be paid to the influence of the resolution of scientific uncertainties on precautionary decisions. The computation of information values is an important issue in this perspective. Furthermore, decisions rules and criteria taking into intergenerational equity requirements should be examined. In particular “maximin” framework could provide adequate alternatives.

A Appendix

A.1 Notations

We need to consider

- The indicatric extended function

$$\Psi(x) = \begin{cases} 0 & \text{if} \qquad x \leq 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{14}$$

- The time dependent value function $V(t,.,.)$ defined for any time $t < T$ by

$$V(t, M, Q) \;\; = \inf_{a_t,\, a_{t+1},\, ..,\, a_{T-1}} \left(\sum_{s=t}^{T-1} \rho^s C(a_s, Q_s) + \Psi(M_T - M^\flat) \right) \tag{15}$$

  and at final period T by $V(T, M, Q) = \Psi(M - M^\flat)$.

- The abatement rate

$$a^+(t, M, Q) = \frac{(1-\delta)(M - M_t^\sharp) + E(Q)}{E(Q)}. \tag{16}$$

- The notation for the concentration dynamics

$$f(M, Q, a) = (1-\delta)(M - M_\infty) + E(Q)(1-a) + M_\infty. \tag{17}$$

We also recall (see Rockafellar and Wets (1998) for instance) that the generalized $\partial_x g(x)$ gradient of a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ at the point x is defined by

$$\partial_x g(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{x' \to x} \frac{g(x') - g(x) - < p, x' - x >}{\|x' - x\|} = 0 \right\}. \tag{18}$$

A.2 Bellman dynamic programming

We obtain

Proposition A.1 *We posit the assumptions (3) and (9). Then there exists a function $\phi(.,.,.)$ such that the optimal value function V satisfies for any $0 \leq t \leq T$ and for any (M, Q)*

$$V(t, M, Q) = \phi(t, M, Q) + \Psi(M - M_t^\sharp), \tag{19}$$

where $M \to \phi(t, M, Q)$ is locally Lipschitz for any (t, Q) and satisfies

$$\partial_M \phi(t, M, Q) \subset \left[0, \frac{c(Q)}{E(Q)}\right].$$

Furthermore an optimal feedback abatement exists and is defined for any $0 \leq t < T$ by

$$a^*(t, M, Q) = \max(0, a^+(t, M, Q)).$$

Proof — of proposition (A.1)
We reason recursively using backward dynamic programming principle.

First, the condition (19) holds true for $t = T$ with

$$\phi(T, M, Q) = 0.$$

Now, assume the condition (19) to hold at time $t + 1$. Using Bellman equation, we can write:

$$\begin{aligned} V(t, M, Q) &= \inf_{a \in [0,1]} \left\{ C(a, Q) + \rho V\Big(t + 1, f(M, Q, a), (1 + g)Q\Big) \right\} \\ &= \inf_{a \in [0,1]} \Big\{ ac(Q) + \rho\phi\Big(t + 1, f(M, Q, a), (1 + g)Q\Big) \\ &\qquad + \Psi\Big(f(M, Q, a) - M_{t+1}^\sharp\Big) \Big\}. \end{aligned}$$

Using definitions of $f(M, Q, a)$ and $M_{t+1}^\sharp$, we check that the condition

$$\Psi\big(f(M, Q, a) - M_{t+1}^\sharp\big) < +\infty$$

is equivalent to claim $aE(Q) \geq f(M, Q, 0) - M_{t+1}^\sharp$. The straightforward computation described in Lemma (A.2) then provides

$$a^+(t, M, Q) \leq a.$$

Combined with condition $a \leq 1$, we derive that $M \leq M_t^\sharp$ or equivalently $\Psi(M - M_t^\sharp) < +\infty$.

Moreover, dynamic programming then reduces to

$$\begin{aligned}
V(t,M,Q) = & \inf_{\substack{a \in [0,1] \\ a \geq a^+(t,M,Q)}} \Bigg\{ C(a,Q) + \rho\phi\Bigg(t+1, f(M,Q,a), (1+g)Q \Bigg) \\
& \qquad + \Psi\Bigg(M - M_t^\sharp \Bigg) \Bigg\} \\
= & \inf_{\substack{a \in [0,1] \\ a \geq a^+(t,M,Q)}} \Bigg\{ \alpha(a) + \Psi\Bigg(M - M_t^\sharp \Bigg) \Bigg\}
\end{aligned}$$

where $\alpha(a) = C(a,Q) + \rho\phi\Bigg(t+1, f(M,Q,a), (1+g)Q \Bigg)$.

Now let us use the recursive condition

$$\partial_M \phi(t+1, M, (1+r)Q) \subset \left[0, \frac{c((1+g)Q)}{E((1+g)Q)} \right].$$

Marginal assumptions on emissions (3) and marginal abatement costs (9) combined with condition $g \geq 0$ yields

$$\partial_M \phi(t+1, M, (1+g)Q) \subset \left[0, \frac{c(Q)}{E(Q)} \right].$$

Thus, using a chain rule, we have

$$\begin{aligned}
\partial_a \alpha(a) & \subset c(Q) + \rho \partial_M \phi(t+1, f(M,Q,a), (1+g)Q) \partial_a f(M,Q,a) \\
& \subset c(Q) + \rho \left[-\tfrac{c(Q)}{E(Q)}, 0 \right] E(Q) \\
& \subset c(Q)[1-\rho, 1] \\
& \subset \mathbb{R}_+^*.
\end{aligned}$$

Therefore, the generalized derivative of the function α is strictly positive and consequently α is increasing with respect to a. Thus the minimal abatement is achieved at the lowest feasible boundary namely

$$\begin{aligned}
a^*(t,Q,M) = & \operatorname{Argmin}_{\left\{ \substack{a \in [0,1] \\ a \geq a^+(t,M,Q)} \right.} \alpha(a) \\
= & \max(0, a^+(t,M,Q)).
\end{aligned}$$

Moreover we set $\phi(t,M,Q) = \alpha(a^*(t,Q,M))$.

It remains to prove that $M \to \phi(t, M, Q)$ is Locally Lipschitz for any (t, Q) and satisfies

$$\partial_M \phi(t, M, Q) \subset \left[0, \frac{c(Q)}{E(Q)}\right].$$

As a maximum of linear functions, it is clear that $M \to a^*(t, M, Q)$ is Lipschitz. The Lipschitziannity of α implies that $M \to \phi(t, M, Q)$ is also Lipschitz.

Now we distinguish two cases:

• If $a^*(t, Q, M) = a^+(t, Q, M)$ then Lemma (A.3) allows to claim $f(M, Q, a^*) = M^\sharp_{t+1}$. We deduce that the value function is

$$\phi(t, M, Q) \quad = \quad a^+(t, M, Q)c(Q) + \rho V(t + 1, M^\sharp_{t+1}, (1 + g)Q).$$

Since $0 \le \delta < 1$, we obtain that

$$\begin{aligned} \partial_M \phi(t, M, Q) \quad &= c(Q)\partial_M a^+(t, M, Q) \\ &= \tfrac{c(Q)(1-\delta)}{E(Q)} \\ &\subset \left[0, \tfrac{c(Q)}{E(Q)}\right] \end{aligned}$$

which is the desired condition.

• Now we consider the case where $a^*(t, Q, M) = 0$. In this case, we obtain

$$\phi(t, M, Q) \quad = C(0, Q) + \rho\phi\Big(t + 1, f(M, Q, 0), (1 + g)Q)\Big)$$

and, using assumptions (3) and (9),

$$\begin{aligned} \partial_M \phi(t, M, Q) \quad &= \rho(1 - \delta)\partial_M \phi(t + 1, f(M, Q, 0), (1 + g)Q) \\ &\subset \rho(1 - \delta)\left[0, \tfrac{c((1+g)Q)}{E((1+g)Q)}\right] \\ &\subset \left[0, \tfrac{c(Q)}{E(Q)}\right]. \end{aligned}$$

This concludes the proof.

A.3 The existence and effectiveness results

Proof — of proposition (3.1)
We use proposition (A.1) and consider the value function $V(t, M_t, Q_t)$ for an optimal trajectory M_t and an optimal feedback $a^*(t, M_t, Q_t)$

$$\left\{ \begin{array}{ll} M_{t+1} & = f(M_t, Q_t, a^*(t, M_t, Q_t)) \\ Q_{t+1} & = (1 + g)Q_t. \end{array} \right.$$

It is well-defined when it takes finite values namely

$$V(t, M_t, Q_t) < +\infty$$

when $\Psi(M_t - M_t^\sharp) = 0$. This means that

$$M_t \leq M_t^\sharp.$$

In particular, for initial time $t = 0$, this yields $M_0 \leq M_0^\sharp$.

A.4 The feedback result

Part of the previous proposition is directly the feedback assertion in proposition (3.2).

A.5 The monotonicity result

Proof — of proposition (3.3)
We need to prove that for every t we have

$$a_t^* \leq a_{t+1}^*.$$

To achieve this, we use Bellman iterative process through the proposition (A.1). We need to distinguish two cases: Indeed

- If the abatement rate is zero *i.e.* $a_t^* = 0$, since $a_{t+1}^* \geq 0$, we easily conclude $a_t^* \leq a_{t+1}^*$.
- If $1 \geq a_t^* > 0$, we deduce by proposition (A.1) that $a_t^* = a^+(t, M_t, Q_t)$. Then Lemma (A.3) yields

  $$M_{t+1} = f(M_t, Q_t, a^+(t, M_t, Q_t)) = M_{t+1}^\sharp.$$

  Therefore by the very definition of a^+, we obtain

  $$a^+(t+1, M_{t+1}^\sharp, Q_{t+1}) = 1,$$

  and consequently

  $$a_{t+1}^* = \max(0, a^+(t+1, M_{t+1}^\sharp, Q_{t+1})) = 1.$$

  We obviously conclude that $a_t^* \leq a_{t+1}^*$.

This ends the proof of proposition (3.3).

A.6 Last time to act

Proof — of proposition (3.4)
The proof proceeds in two steps:

$$\begin{array}{ll} (i) & M_t^{\text{bau}} - M_t^\sharp \leq 0, \ \forall t \leq \text{OTTA}, \\ (ii) & M_t^{\text{bau}} - M_t^\sharp > 0, \ \forall t > \text{OTTA}. \end{array} \tag{20}$$

(i) We consider M_t^* the effective optimal concentration at time t associated with optimal and effective abatement policy a_t^*. From the very definition of $\text{OTTA} = \min(t \geq 0 \mid a_t^* > 0)$, we deduce that

$$\forall t < \text{OTTA}, \ a_t^* = 0 \text{ and } M_{t+1}^* = M_{t+1}^{\text{bau}}.$$

In particular, this shows that $M_{\text{OTTA}}^* = M_{\text{OTTA}}^{\text{bau}}$. Since a_t^* is an effective policy, the concentration M_t^* remains within the tolerable ceiling and we derive that

$$M_{t+1}^{\text{bau}} = M_{t+1}^* \leq M_{t+1}^\sharp, \ \forall t < \text{OTTA}.$$

Consequently condition (20) (i) holds true.

(ii) From the very definition of OTTA, we know that $a_{\text{OTTA}}^* > 0$. Furthermore, since a_t^* increases with time, we have

$$a_t^* = a^*(t, M_t^*, Q_t) > 0, \ \forall t \geq \text{OTTA}.$$

Moreover, the proposition (3.2) allows us to write for every t larger than OTTA

$$a_t^* = a^*(t, M_t^*, Q_t) = \max(0, a^+(t, M_t^*, Q_t)) = a^+(t, M_t^*, Q_t).$$

For any t larger than OTTA, Lemma (A.3) then yields that

$$\begin{aligned} M_{t+1}^* &= f(M_t^*, Q_t, a_t^*) \\ &= f(M_t^*, Q_t, a^+(t, M_t^*, Q_t)) \\ &= M_{t+1}^\sharp. \end{aligned}$$

Since the optimal concentration path M^* is strictly smaller than the BAU path M^{bau} from the optimal time to act, we deduce that

$$M_{t+1}^\sharp < M_{t+1}^{\text{bau}}, \ \forall t \geq \text{OTTA}$$

which is equivalent to condition (20) (ii). We conclude.

A.7 Some useful lemmas

Lemma A.2 *For any* (t, M, Q)*, we have*

$$a^+(t, M, Q) = \frac{f(M, Q, 0) - M^\sharp_{t+1}}{E(Q)}$$

Lemma A.3 *For any* (t, M, Q)*, we have*

$$f(M, Q, a^+(t, M, Q)) = M^\sharp_{t+1}.$$

Proof — of the Lemma (A.2)
We can write

$$\begin{aligned} a^+(t, M, Q)E(Q) \quad &= (1-\delta)(M - M^\sharp_t) + E(Q), \\ &= (1-\delta)(M - M_\infty - (1-\delta)^{t-T}(M^\flat - M_\infty)) + E(Q) \\ &= (1-\delta)(M - M_\infty) - (1-\delta)^{t+1-T}(M^\flat - M_\infty) + E(Q) \\ &= (1-\delta)(M - M_\infty) + E(Q) - (M^\sharp_{t+1} - M_\infty) \\ &= (1-\delta)(M - M_\infty) + E(Q) + M_\infty - M^\sharp_{t+1} \\ &= f(M, Q, 0) - M^\sharp_{t+1}. \end{aligned}$$

This ends the proof.
Proof — of the Lemma (A.3)
We can write

$$\begin{aligned} f(M, Q, a^+(t, M, Q)) \quad &= (1-\delta)(M - M_\infty) + E(Q)(1 - a^+(t, M, Q)) + M_\infty \\ &= f(M, Q, 0) - a^+(t, M, Q)E(Q). \end{aligned}$$

We use (A.2) to deduce

$$f(M, Q, a^+(t, M, Q)) \quad = M^\sharp_{t+1}.$$

which concludes the proof.

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